

# Cramér-Rao bounds for synchronization of rotations

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## Abstract

Synchronization of rotations is the problem of estimating a set of rotations  $R_i \in \text{SO}(n)$ ,  $i = 1 \dots N$  based on noisy measurements of relative rotations  $R_i R_j^\top$ . This fundamental problem has found many recent applications, most importantly in structural biology. We provide a framework to study synchronization as estimation on Riemannian manifolds for arbitrary  $n$  under a large family of noise models. The noise models we address essentially encompass zero-mean isotropic noise, and we develop tools for Gaussian-like as well as heavy-tail types of noise in particular. As a main contribution, we establish formulas for the Fisher information matrix and associated Cramér-Rao bounds of synchronization. We find that these bounds are structured by the pseudoinverse of the measurement graph Laplacian, where edge weights are proportional to measurement quality. We leverage this to provide interpretation and visualization tools for these bounds in both the anchored and anchor-free scenarios. Similar bounds previously established were limited to rotations in the plane and Gaussian-like noise.

Keywords: Synchronization of rotations, estimation on manifolds, estimation on graphs, graph Laplacian, Fisher information, Cramér-Rao bounds, distributions on the rotation group, Langevin.

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## 1 Introduction

Synchronization of rotations is the problem of estimating rotation matrices  $R_1, \dots, R_N$  based on noisy measurements of relative rotations  $R_i R_j^\top$ . The set of available measurements gives rise to a graph structure, where the  $N$  nodes correspond to the rotations  $R_i$  and an edge is present between two nodes  $i$  and  $j$  if a measurement of  $R_i R_j^\top$  is given. Depending on the application, some of the rotations may be known in advance or not. The known rotations, if any, are called *anchors*. In the absence of anchors, it is only possible to recover the rotations up to a global rotation, since the measurements only reveal information about relative rotations. Motivated by the pervasiveness of synchronization of rotations in applications, we propose a derivation and analysis of Cramér-Rao bounds for this estimation problem. Our results hold for rotations in  $\text{SO}(n) = \{R \in \mathbb{R}^{n \times n} : R^\top R = I_n \text{ and } \det R = 1\}$  for arbitrary  $n$  and for a large family of practically useful noise models.

Synchronization of rotations appears naturally in a number of important applications. Tron and Vidal for example consider a network of cameras [37]. Each camera has a certain position in  $\mathbb{R}^3$  and orientation in  $\text{SO}(3)$ . For some pairs of cameras, a calibration procedure produces a

noisy measurement of relative position and relative orientation. The task of using all relative orientation measurements simultaneously to estimate the configuration of the individual cameras is a synchronization problem. Cucuringu et al. address sensor network localization based on inter-node distance measurements [13]. In their approach, they decompose the network in small, overlapping, rigid patches. Each patch is easily embedded in space thanks to its rigidity, but the individual embeddings are noisy. These embeddings are then aggregated by aligning overlapping patches. For each pair of such patches, a measurement of relative orientation is produced. Synchronization permits the use all of these measurements simultaneously to prevent error propagation. In related work, a similar approach is applied to the molecule problem [14]. Tzeneva et al. applied synchronization to the construction of 3D models of objects based on scans of the objects under various unknown orientations [38]. Singer and Shkolnisky study cryo-EM imaging [35]. In this problem, the aim is to produce a 3D model of a macro-molecule based on many projections (pictures) of the molecule under various random and unknown orientations. A procedure specific to the cryo-EM imaging technique helps estimating the relative orientation between pairs of projections, but this process is very noisy. In fact, most measurements are outliers. The task is to use these noisy measurements of relative orientations of images to recover the true orientations under which the images were acquired. This naturally falls into the scope of synchronization of rotations, and calls for very robust algorithms.

### 1.1 Previous work

Tron and Vidal, in their work about camera calibration, develop distributed algorithms based on consensus on manifolds to solve synchronization on  $\mathbb{R}^3 \times \text{SO}(3)$  [37]. Singer studies synchronization of phases, that is, rotations in the plane, and reflects upon the generic nature of synchronization as the task of estimating group elements  $g_1, \dots, g_N$  based on measurements of their ratios  $g_i g_j^{-1}$  [34]. In that work, the author focuses on synchronization in the presence of many outliers in the measurements and establishes that synchronization is surprisingly robust against such outliers. Bounds are derived based on information-theoretic arguments to establish how many measurements need to be accurate for synchronization to be possible. A fast algorithm based on eigenvector computations is shown to obtain good solutions. In particular, it performs better than random estimation as soon as the information-theoretic threshold is reached. In further work, Bandeira et al. go deeper in the analysis and generalize the bounds and the algorithms to rotations in  $\mathbb{R}^n$  [4]. Hartley et al. develop a robust algorithm based on a Weiszfeld iteration to compute L1-means on  $\text{SO}(n)$ , and extend the algorithm to perform robust synchronization of rotations [18]. Russel et al. develop a decentralized algorithm for synchronization on the group of translations  $\mathbb{R}^n$  [31]. Barooah and Hespanha study the covariance of the BLUE estimator for synchronization on  $\mathbb{R}^n$  with anchors. This covariance coincides with the Cramér-Rao bound (CRB) under Gaussian noise. They give interpretations of the covariance in terms of the resistance distance on the measurement graph [5]. Howard et al. study synchronization on the group of translations  $\mathbb{R}^n$  and on the group of phases  $\text{SO}(2)$  [21]. They establish CRB's for synchronization in the presence of Gaussian-like noise on these groups and provide decentralized algorithms to solve synchronization. Their derivation of the CRB's seems to rely heavily on the commutativity (and thus flatness) of  $\mathbb{R}^n$  and  $\text{SO}(2)$ , and hence does not apply to synchronization on  $\text{SO}(n)$  in general. Furthermore, they only analyze Gaussian-like noise.

CRB's are a classical tool in estimation theory [29] that provide a lower-bound on the variance of any unbiased estimator for an estimation problem, based on the measurements' distribution. The classical results focus on estimation problems for which the sought parameter belongs to a Euclidean space. In the context of synchronization, this is not sufficient, since the parameters we seek belong to the manifold of rotations  $\text{SO}(n)$ . Important work by Smith [36] as well as by Xavier and Barroso [41] extends the theory of CRB's to the realm of manifolds in a practical way. Recall that, in the absence of anchors, synchronization can only be solved up to a global rotation. This ambiguity, as we shall see, gives rise to a singular Fisher information matrix (FIM). As the CRB's are usually expressed in terms of the inverse of the FIM, this is problematic. Xavier and Barroso provide a nice geometric interpretation in terms of estimation on quotient manifolds of the well-known fact that one may use the pseudoinverse of the FIM in such situations [40]. It is then apparent that to establish CRB's for synchronization, one needs to either impose anchors,

leading to a submanifold geometry, or work on the quotient manifold. In [7], an attempt is made at providing a unified framework to build such CRB's. We use these tools in the present work.

Other authors have established CRB's for sensor network localization (SNL) and synchronization problems. Ash and Moses study SNL based on inter-agent distance measurements, and notably give an interpretation of the CRB in the absence of anchors [3]. Chang and Sahai tackle the same problem [10]. As we mentioned earlier, Howard et al. derive CRB's for synchronization on  $\mathbb{R}^n$  and  $\text{SO}(2)$  [21]. CRB's for synchronization on  $\mathbb{R}^n$  are re-derived as an example in [7]. A remarkable fact is that, for all these problems of estimation on graphs, the pseudoinverse of the graph Laplacian plays a fundamental role in the CRB—although not all authors explicitly reflect on this. As we shall see, this special structure is rich in interpretations, many of which exceed the context of synchronization of rotations specifically.

## 1.2 Contributions and outline

In this work, we state the problem of synchronization of rotations in  $\text{SO}(n)$  for arbitrary  $n$  as an estimation problem on a manifold—Section 2. We describe the geometry of this manifold both for anchored synchronization (giving rise to a submanifold geometry) and for anchor-free synchronization (giving rise to a quotient geometry)—Section 3. Among other things, this paves the way for maximum-likelihood estimation using optimization on manifolds [1], which will be the focus of further work.

We describe a family of noise models (probability density functions) on  $\text{SO}(n)$  that fulfill a few assumptions—Section 4. We show that this family is both useful for applications (it essentially contains zero-mean, isotropic noise models) and practical to work with (the expectations one is lead to compute via integrals on  $\text{SO}(n)$  are easily converted into classical integrals on  $\mathbb{R}^n$ ). In particular, this family includes a kind of heavy-tailed distribution on  $\text{SO}(n)$  that appears to be new. We describe this distribution and we are convinced it will prove useful for other estimation problems on  $\text{SO}(n)$  with outliers.

In Section 5, we derive the Fisher information metric (FIM) for synchronization and establish it is structured by the Laplacian of the measurement graph, where edge weights are proportional to the quality of their respective measurements. The FIM plays a central role in the Cramér-Rao bounds (CRB) we establish for anchored and anchor-free synchronization—Section 6. The CRB's are structured by the pseudoinverse of the Laplacian of the measurement graph. We derive clear interpretations of these bounds in both cases (with visualization tools), and note they differ significantly.

As a main result for anchored synchronization, we show that for any unbiased estimator  $\hat{R}_i$  of the rotation  $R_i$ , asymptotically for small errors,

$$\mathbb{E} \left\{ \text{dist}^2(R_i, \hat{R}_i) \right\} \geq d^2 (\mathcal{L}_A^\dagger)_{ii}, \quad (1.1)$$

where  $\text{dist}$  is the geodesic distance on  $\text{SO}(n)$ ,  $d = n(n-1)/2$ ,  $\mathcal{L}_A$  is the Laplacian of the weighted measurement graph with rows and columns corresponding to anchors set to zero and  $\dagger$  denotes the Moore-Penrose pseudoinverse—see (6.10). The better a measurement is, the larger the weight on the associated edge is—see (5.20). This bound holds in a small-error regime under the assumption that noise on different measurements is independent, that the measurements are isotropically distributed around the true relative rotations and that there is at least one anchor in each connected component of the graph.

As a main result for anchor-free synchronization, we show that for any unbiased estimator  $\hat{R}_i \hat{R}_j^\top$  of the relative rotation  $R_i R_j^\top$ , asymptotically for small errors,

$$\mathbb{E} \left\{ \text{dist}^2(R_i R_j^\top, \hat{R}_i \hat{R}_j^\top) \right\} \geq d^2 (e_i - e_j)^\top \mathcal{L}^\dagger (e_i - e_j), \quad (1.2)$$

where  $\mathcal{L}$  is the Laplacian of the weighted measurement graph and  $e_i$  is the  $i^{\text{th}}$  column of the  $N \times N$  identity matrix—see (6.24). This bound holds in a small-error regime under the assumption that noise on different measurements is independent, that the measurements are isotropically distributed around the true relative rotations and that the measurement graph is connected.

Section 7 hosts a few comments on the CRB's. In particular, a link with the Fiedler value of the graph is described and the robustness of synchronization versus outliers is confirmed, via arguments that differ from those in [34].

## 2 Synchronization of rotations

Synchronization of rotations is the problem of estimating a set of rotations  $R_1, \dots, R_N \in \text{SO}(n)$  from noisy measurements of some of the relative rotations  $R_i R_j^\top$ . In this section, we model synchronization as an estimation problem on a manifold, since the set of rotations  $\text{SO}(n)$  is a Lie group, i.e., a group and a manifold at the same time.

In our estimation problem, the target quantities (the parameters) are the rotation matrices  $R_1, \dots, R_N \in \text{SO}(n)$ . The natural parameter space is thus:

$$\mathcal{P} = \text{SO}(n) \times \dots \times \text{SO}(n) \quad (N \text{ copies}). \quad (2.1)$$

Let  $[N] \triangleq \{1, \dots, N\}$ . Consider a set  $\mathcal{E} \subset [N] \times [N]$  such that  $(i, j) \in \mathcal{E} \Rightarrow i \neq j$  and  $(j, i) \in \mathcal{E}$ . This set defines an undirected graph over  $N$  nodes,

$$G = ([N], \mathcal{E}) \quad (\text{the measurement graph}). \quad (2.2)$$

For each edge  $(i, j) \in \mathcal{E}$ ,  $i < j$ , we have a measurement  $H_{ij} \in \text{SO}(n)$  of the form

$$H_{ij} = Z_{ij} R_i R_j^\top, \quad (2.3)$$

where  $Z_{ij}$  is a random variable distributed over  $\text{SO}(n)$  following a probability density function (pdf)  $f_{ij} : \text{SO}(n) \rightarrow \mathbb{R}^+$ , with respect to the Haar measure  $\mu$  on  $\text{SO}(n)$ —see Section 4. For example, when  $Z_{ij}$  is deterministically equal to the identity matrix  $I$ , the measurement is perfect (noise-free), whereas when  $Z_{ij}$  is uniformly distributed over (the compact set)  $\text{SO}(n)$ , the measurement contains no information. We say that the measurement is *unbiased*, or that the noise has *zero-mean*, if the mean value of  $Z_{ij}$  is the identity matrix—a notion we make precise in Section 4. We also say that noise is *isotropic* if its probability density function is only a function of distance to the identity. Different notions of distance on  $\text{SO}(n)$  yield different notions of isotropy. In Section 4 we give a few examples of useful zero-mean, isotropic distributions on  $\text{SO}(n)$ .

Pairs  $(i, j)$  and  $(j, i)$  in  $\mathcal{E}$  refer to the same measurement. By symmetry, for  $i < j$ , we define  $H_{ji} = Z_{ji} R_j R_i^\top = H_{ij}^\top$  and the random variable  $Z_{ji}$  and its density  $f_{ji}$  are defined accordingly in terms of  $f_{ij}$  and  $Z_{ij}$ . In particular,

$$Z_{ji} = R_j R_i^\top Z_{ij}^\top R_i R_j^\top, \quad \text{and} \quad f_{ij}(Z_{ij}) = f_{ji}(Z_{ji}). \quad (2.4)$$

The pdf's  $f_{ij}$  and  $f_{ji}$  are linked as such because the Haar measure  $\mu$  is invariant under the change of variable relating  $Z_{ij}$  and  $Z_{ji}$ —see Section 4.

In this work, we restrict our attention to noise models that fulfill the three following assumptions:

**Assumption 1** (smoothness and support). *Each pdf  $f_{ij} : \text{SO}(n) \rightarrow \mathbb{R}^+ = (0, +\infty)$  is a smooth, positive function.*

**Assumption 2** (independence). *The  $Z_{ij}$ 's associated to different edges of the measurement graph are independent random variables. That is, if  $(i, j) \neq (p, q)$  and  $(i, j) \neq (q, p)$ ,  $Z_{ij}$  and  $Z_{pq}$  are independent.*

**Assumption 3** (invariance). *Each pdf  $f_{ij}$  is invariant under orthogonal conjugation, that is,  $\forall Z \in \text{SO}(n), \forall Q \in \text{O}(n), f_{ij}(Q Z Q^\top) = f_{ij}(Z)$ . We say  $f_{ij}$  is a spectral function, since it only depends on the eigenvalues of its argument. The eigenvalues of matrices in  $\text{SO}(2k)$  have the form  $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_k}$ , with  $0 \leq \theta_1, \dots, \theta_k \leq \pi$ . The eigenvalues of matrices in  $\text{SO}(2k+1)$  have an additional eigenvalue 1.*

Assumption 1 is satisfied for all the noise models we consider; it could be relaxed to some extent but would make some of the proofs more technical. Assumption 2 is necessary to make the joint pdf of the whole estimation problem easy to derive, leading to an easy expression for the log-likelihood function. As we will see in Section 5, it is also at the heart of the nice Laplacian structure of the Fisher information matrix. Assumption 3 is a technical condition that will prove useful in many respects. One of them is the observation that pdf's that obey Assumption 3 are easy to integrate over  $\text{SO}(n)$ . We expand on this in Section 4, where we also show that a large family of interesting pdf's satisfy these assumptions, namely, zero-mean isotropic distributions.

Under Assumption 2, the *log-likelihood* of an estimator  $\hat{\mathbf{R}} = (\hat{R}_1, \dots, \hat{R}_N) \in \mathcal{P}$ , given the measurements  $H_{ij}$ , is given by:

$$L(\hat{\mathbf{R}}) = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} \log f_{ij}(H_{ij} \hat{R}_j \hat{R}_i^\top) = \frac{1}{2} \sum_{i=1}^N \sum_{j \in V_i} \log f_{ij}(H_{ij} \hat{R}_j \hat{R}_i^\top), \quad (2.5)$$

where  $V_i \subset [N]$  is the set of neighbors of node  $i$ , i.e.,  $j \in V_i \Leftrightarrow (i, j) \in \mathcal{E}$ . The coefficient  $1/2$  reflects the fact that measurements  $(i, j)$  and  $(j, i)$  give the same information and are deterministically linked. Under Assumption 1,  $L$  is a smooth function on the smooth manifold  $\mathcal{P}$ .

The log-likelihood function is invariant under a global rotation. Indeed,

$$\forall \hat{\mathbf{R}} \in \mathcal{P}, \forall Q \in \text{SO}(n), \quad L(\hat{\mathbf{R}}Q) = L(\hat{\mathbf{R}}), \quad (2.6)$$

where  $\hat{\mathbf{R}}Q$  denotes  $(\hat{R}_1Q, \dots, \hat{R}_NQ) \in \mathcal{P}$ . This invariance encodes the fact that all sets of rotations of the form  $\hat{\mathbf{R}}Q$  for  $Q \in \text{SO}(n)$  yield the same distribution of the measurements  $H_{ij}$ , and are hence indistinguishable, i.e., rotations can only be recovered up to a global rotation.

To resolve the ambiguity, one can follow at least two courses of action. One is to include additional constraints, most naturally in the form of anchors, i.e., assume some of the rotations are known<sup>1</sup>. The other is to acknowledge the invariance by working on the associated quotient space.

Following the first path, the parameter space becomes a Riemannian submanifold of  $\mathcal{P}$ . Following the second path, the parameter space becomes a Riemannian quotient manifold of  $\mathcal{P}$ . In the next section, we describe the geometry of both. In Section 4, we review useful tools to compute integrals on  $\text{SO}(n)$  and describe probability density functions that are both useful to model practical problems and fulfill our assumptions. In Section 5 we use the expression of the log-likelihood function (2.5) to derive the Fisher information matrix for synchronization, which plays a central role in deriving the Cramér-Rao bounds for synchronization with and without anchors—see Section 6.

**Remark 1** (A word about other noise models). *We show that measurements of the form  $H_{ij} = Z_{ij,1} R_i R_j^\top Z_{ij,2}$ , with  $Z_{ij,1}$  and  $Z_{ij,2}$  two independent random rotations with pdf's satisfying Assumptions 1–3, satisfy the noise model considered in the present work. In doing so, we use some material from Section 4. For notational convenience, let us consider  $H = Z_1 R Z_2$ , with  $Z_1, Z_2$  two independent random rotations with pdf's  $f_1, f_2$  satisfying Assumption 1–3,  $R \in \text{SO}(n)$  fixed. Then, the pdf of  $H$  is the function  $h : \text{SO}(n) \rightarrow \mathbb{R}^+$  given by (essentially) the convolution of  $f_1$  and  $f_2$  on  $\text{SO}(n)$ :*

$$h(H) = \int_{\text{SO}(n)} f_1(Z) f_2(R^\top Z^\top H) d\mu(Z) = \int_{\text{SO}(n)} f_1(Z) f_2(Z^\top H R^\top) d\mu(Z), \quad (2.7)$$

where we used that  $f_2$  is spectral:  $f_2(R^\top Z^\top H) = f_2(R R^\top Z^\top H R^\top)$ . Let  $Z_{\text{eq}}$  be a random rotation with smooth pdf  $f_{\text{eq}}$ . We will shape  $f_{\text{eq}}$  such that the random rotation  $Z_{\text{eq}} R$  has the same distribution as  $H$ . This condition can be written as follows: for all measurable subsets  $S \subset \text{SO}(n)$ ,

$$\int_S h(Z) d\mu(Z) = \int_{SR^\top} f_{\text{eq}}(Z) d\mu(Z) = \int_S f_{\text{eq}}(ZR^\top) d\mu(Z), \quad (2.8)$$

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<sup>1</sup>If we only know that  $R_i$  is close to some matrix  $\bar{R}$ , and not necessarily equal to it, we may add a phony node  $R_{N+1}$  anchored at  $\bar{R}$ , and link that node and  $R_i$  with a high confidence measure  $H_{i,N+1} = I_n$ . This makes it possible to have “soft anchors”.

where, going from the second to the third integral, we used the change of variable  $Z := ZR^\top$  and the bi-invariance of the Haar measure  $\mu$ . In words: for all  $S$ , the probability that  $H$  belongs to  $S$  must be the same as the probability that  $Z_{\text{eq}}R$  belongs to  $S$ . This must hold for all  $S$ , hence  $f_{\text{eq}}(Z_{\text{eq}}R^\top) = h(Z_{\text{eq}})$ , or equivalently:

$$f_{\text{eq}}(Z_{\text{eq}}) = h(Z_{\text{eq}}R) = \int_{\text{SO}(n)} f_1(Z) f_2(Z^\top Z_{\text{eq}}) d\mu(Z). \quad (2.9)$$

This uniquely defines the pdf of  $Z_{\text{eq}}$ . It remains to show that  $f_{\text{eq}}$  is a spectral function. For all  $Q \in \text{O}(n)$ ,

$$f_{\text{eq}}(QZ_{\text{eq}}Q^\top) = \int_{\text{SO}(n)} f_1(Z) f_2(Z^\top QZ_{\text{eq}}Q^\top) d\mu(Z) \quad (2.10)$$

$$(f_2 \text{ is spectral}) = \int_{\text{SO}(n)} f_1(Z) f_2(Q^\top Z^\top QZ_{\text{eq}}) d\mu(Z) \quad (2.11)$$

$$(\text{change of variable: } Z := QZQ^\top) = \int_{\text{SO}(n)} f_1(QZQ^\top) f_2(Z^\top Z_{\text{eq}}) d\mu(Z) \quad (2.12)$$

$$(f_1 \text{ is spectral}) = \int_{\text{SO}(n)} f_1(Z) f_2(Z^\top Z_{\text{eq}}) d\mu(Z) = f_{\text{eq}}(Z_{\text{eq}}). \quad (2.13)$$

Hence, the noise model  $H_{ij} = Z_{ij,1}R_iR_j^\top Z_{ij,2}$  can be replaced with the model  $H_{ij} = Z_{ij,\text{eq}}R_iR_j^\top$  and the pdf of  $Z_{ij,\text{eq}}$  is such that it falls within the scope of the present work.

In particular, if  $f_1$  is a point mass at the identity, so that  $H = RZ_2$  (noise multiplying the relative rotation on the right rather than on the left),  $f_{\text{eq}} = f_2$ , so that it does not matter whether we consider  $H_{ij} = Z_{ij}R_iR_j^\top$  or  $H_{ij} = R_iR_j^\top Z_{ij}$ : they have the same distribution.

### 3 Geometry of the parameter spaces

We start with a quick reminder of the geometry of  $\text{SO}(n)$ . It is assumed that the reader is familiar with standard concepts from Riemannian geometry [1, 6, 28]. We then go on to describe the parameter spaces for the anchored and the anchor-free cases of synchronization.

The group of rotations  $\text{SO}(n) = \{Q \in \mathbb{R}^{n \times n} : Q^\top Q = I \text{ and } \det Q = 1\}$  is a connected, compact Lie group. Its Lie algebra is the set of skew-symmetric matrices:

$$\mathfrak{so}(n) = \{\Omega \in \mathbb{R}^{n \times n} : \Omega + \Omega^\top = 0\}. \quad (3.1)$$

This is also the tangent space at the identity  $T_I \text{SO}(n) = \mathfrak{so}(n)$ . The tangent space at  $Q \in \text{SO}(n)$  is linked to  $\mathfrak{so}(n)$  via:

$$T_Q \text{SO}(n) = Q\mathfrak{so}(n) = \{Q\Omega : \Omega \in \mathfrak{so}(n)\}. \quad (3.2)$$

As usual, we consider  $\text{SO}(n)$  as a Riemannian submanifold of the general linear group, and hence endow it with the usual metric:

$$\langle Q\Omega_1, Q\Omega_2 \rangle_Q = \langle \Omega_1, \Omega_2 \rangle = \text{trace}(\Omega_1^\top \Omega_2), \quad \|Q\Omega\|_Q^2 = \langle Q\Omega, Q\Omega \rangle_Q = \|\Omega\|_{\mathbb{F}}^2. \quad (3.3)$$

For greater readability, we often omit the subscripts  $Q$ . The orthogonal projector from the embedding space  $\mathbb{R}^{n \times n}$  onto the tangent space  $T_Q \text{SO}(n)$  is:

$$P_Q(H) = Q \text{sk}(Q^\top H), \text{ with } \text{sk}(A) \triangleq (A - A^\top)/2. \quad (3.4)$$

The exponential map and the logarithmic map accept simple expressions in terms of matrix exponential and logarithm:

$$\text{Exp}_Q(Q\Omega) = Q \exp(\Omega), \quad \text{Log}_{Q_1}(Q_2) = Q_1 \log(Q_1^\top Q_2). \quad (3.5)$$

The mapping  $t \mapsto \text{Exp}_Q(tQ\Omega)$  defines a geodesic curve on  $\text{SO}(n)$ , passing through  $Q$  with velocity  $Q\Omega$  at time  $t = 0$ . Geodesic curves have zero acceleration and may be considered as the equivalent



of straight lines on manifolds. The logarithmic map  $\text{Log}_Q$  is (locally) the inverse of the exponential map  $\text{Exp}_Q$ . The Riemannian (or geodesic) distance on  $\text{SO}(n)$  is the length of the shortest path (the geodesic arc) joining two points:

$$\text{dist}(Q_1, Q_2) = \|\text{Log}_{Q_1}(Q_2)\|_{Q_1} = \|\log(Q_1^\top Q_2)\|_{\text{F}}. \quad (3.6)$$

In particular, for rotations in the plane ( $n = 2$ ) and in space ( $n = 3$ ), the Riemannian distance between  $Q_1$  and  $Q_2$  is  $\sqrt{2}\theta$ , where  $\theta \in [0, \pi]$  is the angle by which  $Q_1^\top Q_2$  rotates.

Let  $\tilde{f} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$  be a differentiable function, and let  $f = \tilde{f}|_{\text{SO}(n)}$  be its restriction to  $\text{SO}(n)$ . The gradient of  $f$  is a tangent vector field to  $\text{SO}(n)$  uniquely defined by:

$$\langle \text{grad} f(Q), Q\Omega \rangle = \text{D}f(Q)[Q\Omega] \quad \forall \Omega \in \mathfrak{so}(n), \quad (3.7)$$

with  $\text{grad} f(Q) \in \text{T}_Q \text{SO}(n)$  and  $\text{D}f(Q)[Q\Omega]$  the directional derivative of  $f$  at  $Q$  along  $Q\Omega$ . Let  $\nabla \tilde{f}(Q)$  be the usual gradient of  $\tilde{f}$  in  $\mathbb{R}^{n \times n}$ . Then, the gradient of  $f$  is easily computed as [1, eq. (3.37)]:

$$\text{grad} f(Q) = \text{P}_Q(\nabla \tilde{f}(Q)). \quad (3.8)$$

In the sequel, we often write  $\nabla f$  to denote the gradient of  $f$  seen as a function in  $\mathbb{R}^{n \times n}$ , even if it is defined on  $\text{SO}(n)$ .

The parent parameter space for synchronization is the product Lie group  $\mathcal{P} = \text{SO}(n)^N$ . Its geometry is trivially obtained by element-wise extension of the geometry of  $\text{SO}(n)$  just described. In particular, tangent spaces and the Riemannian metric are given by:

$$\text{T}_{\mathbf{R}} \mathcal{P} = \{\mathbf{R}\Omega = (R_1\Omega_1, \dots, R_N\Omega_N) : \Omega_1, \dots, \Omega_N \in \mathfrak{so}(n)\}, \quad (3.9)$$

$$\langle \mathbf{R}\Omega, \mathbf{R}\Omega' \rangle_{\mathbf{R}} = \sum_{i=1}^N \text{trace}(\Omega_i^\top \Omega'_i). \quad (3.10)$$

The gradient of the log-likelihood function  $L$  (2.5),  $\text{grad} L(\hat{\mathbf{R}})$ , is a tangent vector in  $\text{T}_{\hat{\mathbf{R}}} \mathcal{P}$ . The  $i^{\text{th}}$  component of this gradient, that is, the gradient of the mapping  $\hat{R}_i \mapsto L(\hat{\mathbf{R}})$  with  $\hat{R}_{j \neq i}$  fixed, is a vector field on  $\text{SO}(n)$  which can be written as:

$$\text{grad}_i L(\hat{\mathbf{R}}) = \sum_{j \in V_i} \text{grad} \left( X \mapsto \log f_{ij}(H_{ij} \hat{R}_j X^\top) \right) (\hat{R}_i) \quad (3.11)$$

$$= \sum_{j \in V_i} \frac{1}{f_{ij}(H_{ij} \hat{R}_j \hat{R}_i^\top)} \text{grad} \left( X \mapsto f_{ij}(H_{ij} \hat{R}_j X^\top) \right) (\hat{R}_i) \quad (3.12)$$

$$= \sum_{j \in V_i} \left[ \text{grad} \log f_{ij}(H_{ij} \hat{R}_j \hat{R}_i^\top) \right]^\top H_{ij} \hat{R}_j. \quad (3.13)$$

### 3.1 Anchored case

In specific applications, we may know some of the rotation matrices  $R_i$ . Let  $A \subset [N]$  be the set of indices of known rotations, called *anchors*. The associated parameter space

$$\mathcal{P}_A = \{\hat{\mathbf{R}} = (\hat{R}_1, \dots, \hat{R}_N) \in \mathcal{P} : \forall i \in A, \hat{R}_i = R_i\} \quad (3.14)$$

is a Riemannian submanifold of  $\mathcal{P}$ . The tangent space at  $\hat{\mathbf{R}} \in \mathcal{P}_A$  is given by:

$$\text{T}_{\hat{\mathbf{R}}} \mathcal{P}_A = \{H \in \text{T}_{\hat{\mathbf{R}}} \mathcal{P} : \forall i \in A, H_i = 0\}, \quad (3.15)$$

such that the orthogonal projector  $\text{P}_{\hat{\mathbf{R}}} : \text{T}_{\hat{\mathbf{R}}} \mathcal{P} \rightarrow \text{T}_{\hat{\mathbf{R}}} \mathcal{P}_A$  simply sets to zero all components of a tangent vector that correspond to anchored rotations. All tools on  $\mathcal{P}_A$  (exponential and logarithmic map for example) are inherited in the obvious fashion from  $\mathcal{P}$ . In particular, the Riemannian distance on  $\mathcal{P}_A$  is:

$$\text{dist}^2(\hat{\mathbf{R}}, \hat{\mathbf{R}}') = \sum_{i \notin A} \left\| \log(\hat{R}_i^\top \hat{R}'_i) \right\|_{\text{F}}^2. \quad (3.16)$$

### 3.2 Anchor-free case

When no anchors are provided, the rotations can only be recovered up to a common rotation, since the measurements only provide information regarding relative rotations. More precisely, the distribution of the measurements  $H_{ij}$  (2.3) is the same whether the true rotations are  $(R_1, \dots, R_N)$  or  $(R_1Q, \dots, R_NQ)$ , regardless of  $Q \in \text{SO}(n)$ . Consequently, the measurements contain no information as to which of those sets of rotations is the right one. This leads us to define the equivalence relation

$$(R_1, \dots, R_N) \sim (R'_1, \dots, R'_N) \iff \exists Q \in \text{SO}(n) : R_i = R'_i Q \text{ for } i = 1, \dots, N. \quad (3.17)$$

This equivalence relation partitions  $\mathcal{P}$  into equivalence classes, often called *fibers*. The quotient space (the set of equivalence classes)

$$\mathcal{P}_\emptyset \triangleq \mathcal{P} / \sim, \quad (3.18)$$

is again a smooth manifold (in fact,  $\mathcal{P}_\emptyset$  is a *coset manifold* because it results from the quotient of the Lie group  $\mathcal{P}$  by a closed subgroup of  $\mathcal{P}$  [28, Prop. 11.12]). The notation reminds us that the set of anchors  $A$  is empty. Naturally, the log-likelihood function  $L$  is constant over equivalence classes and hence descends as a well-defined function on  $\mathcal{P}_\emptyset$ . Each fiber

$$[\mathbf{R}] = \{(R_1Q, \dots, R_NQ) : Q \in \text{SO}(n)\} \in \mathcal{P}_\emptyset \quad (3.19)$$

is a Riemannian submanifold of the total space  $\mathcal{P}$ . As such, at each point  $\mathbf{R}$ , the fiber  $[\mathbf{R}]$  accepts a tangent space that is a subspace of  $\text{T}_{\mathbf{R}}\mathcal{P}$ . That tangent space to the fiber is called the *vertical space* at  $\mathbf{R}$ , noted  $\text{V}_{\mathbf{R}}$ . Vertical vectors point along directions that are parallel to the fibers. Vectors orthogonal, in the sense of the Riemannian metric (3.10), to all vertical vectors form the *horizontal space*  $\text{H}_{\mathbf{R}} = (\text{V}_{\mathbf{R}})^\perp$ , such that the tangent space  $\text{T}_{\mathbf{R}}\mathcal{P}$  is equal to the direct sum  $\text{V}_{\mathbf{R}} \oplus \text{H}_{\mathbf{R}}$ . Horizontal vectors are orthogonal to the fibers, hence point toward the other fibers, i.e., the other points on the quotient space  $\mathcal{P}_\emptyset$ .

Because  $\mathcal{P}_\emptyset$  is a coset manifold, the projection

$$\pi : \mathcal{P} \rightarrow \mathcal{P}_\emptyset : \mathbf{R} \mapsto \pi(\mathbf{R}) = [\mathbf{R}] \quad (3.20)$$

is a *submersion*. That is, the restricted differential  $D\pi|_{\text{H}_{\mathbf{R}}}$  is a full-rank linear map between  $\text{H}_{\mathbf{R}}$  and  $\text{T}_{[\mathbf{R}]} \mathcal{P}_\emptyset$ . Practically, this means that the horizontal space  $\text{H}_{\mathbf{R}}$  is naturally identified to the (abstract) tangent space  $\text{T}_{[\mathbf{R}]} \mathcal{P}_\emptyset$ . This results in a practical means of representing abstract vectors of  $\text{T}_{[\mathbf{R}]} \mathcal{P}_\emptyset$  simply as vectors of  $\text{H}_{\mathbf{R}} \subset \text{T}_{\mathbf{R}}\mathcal{P}$ , where  $\mathbf{R}$  is any arbitrarily chosen member of  $[\mathbf{R}]$ . Each horizontal vector  $\xi_{\mathbf{R}}$  is unambiguously related to its abstract counterpart  $\xi_{[\mathbf{R}]}$  in  $\text{T}_{[\mathbf{R}]} \mathcal{P}_\emptyset$  via

$$\xi_{[\mathbf{R}]} = D\pi(\mathbf{R})[\xi_{\mathbf{R}}]. \quad (3.21)$$

The representation  $\xi_{\mathbf{R}}$  of  $\xi_{[\mathbf{R}]}$  is called the *horizontal lift* of  $\xi_{[\mathbf{R}]}$  at  $\mathbf{R}$ , a notion made precise in [1, § 3.5.8] and depicted for intuition in [40, Fig. 1–2]—these figures are also reproduced in [7, Fig. 2–3].

Consider  $\xi_{[\mathbf{R}]}$  and  $\eta_{[\mathbf{R}]}$ , two tangent vectors at  $[\mathbf{R}]$ . Let  $\xi_{\mathbf{R}}$  and  $\eta_{\mathbf{R}}$  be their horizontal lifts at  $\mathbf{R} \in [\mathbf{R}]$  and let  $\xi_{\mathbf{R}'}$  and  $\eta_{\mathbf{R}'}$  be their horizontal lifts at  $\mathbf{R}' \in [\mathbf{R}]$ . The Riemannian metric on  $\mathcal{P}$  (3.10) is such that  $\langle \xi_{\mathbf{R}}, \eta_{\mathbf{R}} \rangle_{\mathbf{R}} = \langle \xi_{\mathbf{R}'}, \eta_{\mathbf{R}'} \rangle_{\mathbf{R}'}$ . This motivates us to define the metric

$$\langle \xi_{[\mathbf{R}]}, \eta_{[\mathbf{R}]} \rangle_{[\mathbf{R}]} = \langle \xi_{\mathbf{R}}, \eta_{\mathbf{R}} \rangle_{\mathbf{R}} \quad (3.22)$$

on  $\mathcal{P}_\emptyset$ , which is then well defined (it does not depend on our choice of  $\mathbf{R}$  in  $[\mathbf{R}]$ ) and turns the restricted differential  $D\pi(\mathbf{R}) : \text{H}_{\mathbf{R}} \rightarrow \text{T}_{[\mathbf{R}]} \mathcal{P}_\emptyset$  into an isometry. This is a Riemannian metric and it is the only such metric such that  $\pi$  (3.20) is a *Riemannian submersion* from  $\mathcal{P}$  to  $\mathcal{P}_\emptyset$  [17, Prop. 2.28]. Hence,  $\mathcal{P}_\emptyset$  is a *Riemannian quotient manifold* of  $\mathcal{P}$ .

We now describe the vertical and horizontal spaces of  $\mathcal{P}$  w.r.t. the equivalence relation (3.17). Let  $\mathbf{R} \in \mathcal{P}$  and  $Q : \mathbb{R} \rightarrow \text{SO}(n) : t \mapsto Q(t)$  such that  $Q$  is smooth and  $Q(0) = I$ . Then, the derivative  $Q'(0) = \Omega$  is some skew-symmetric matrix in  $\mathfrak{so}(n)$ . Since  $(R_1Q(t), \dots, R_NQ(t)) \in [\mathbf{R}]$  for all  $t$ , it follows that  $\frac{d}{dt} (R_1Q(t), \dots, R_NQ(t))|_{t=0} = (R_1\Omega, \dots, R_N\Omega)$  is a tangent vector to the fiber  $[\mathbf{R}]$  at  $\mathbf{R}$ , i.e., it is a vertical vector at  $\mathbf{R}$ . All vertical vectors have such form, hence:

$$\text{V}_{\mathbf{R}} = \{(R_1\Omega, \dots, R_N\Omega) : \Omega \in \mathfrak{so}(n)\}. \quad (3.23)$$



A horizontal vector  $(R_1\Omega_1, \dots, R_N\Omega_N) \in \mathbf{H}_{\mathbf{R}}$  is orthogonal to all vertical vectors, i.e.,  $\forall \Omega \in \mathfrak{so}(n)$ ,

$$0 = \langle (R_1\Omega_1, \dots, R_N\Omega_N), (R_1\Omega, \dots, R_N\Omega) \rangle = \sum_{i=1}^N \text{trace}((R_i\Omega_i)^\top R_i\Omega) = \left\langle \sum_{i=1}^N \Omega_i, \Omega \right\rangle. \quad (3.24)$$

Since this is true for all skew-symmetric matrices  $\Omega$ , we find that the horizontal space is defined as:

$$\mathbf{H}_{\mathbf{R}} = \{(R_1\Omega_1, \dots, R_N\Omega_N) : \Omega_1, \dots, \Omega_N \in \mathfrak{so}(n) \text{ and } \sum_{i=1}^N \Omega_i = 0\}. \quad (3.25)$$

This is not surprising: vertical vectors move all rotations in the same direction, remaining in the same equivalence class, whereas horizontal vectors move away toward other equivalence classes.

We now define the logarithmic map on  $\mathcal{P}_\emptyset$ . Considering two points  $[\mathbf{R}], [\hat{\mathbf{R}}] \in \mathcal{P}_\emptyset$ , the logarithm  $\text{Log}_{[\mathbf{R}]}([\hat{\mathbf{R}}])$  is the smallest tangent vector in  $\mathbf{T}_{[\mathbf{R}]} \mathcal{P}_\emptyset$  that brings us from the first equivalence class to the other through the exponential map. Working with the horizontal lift representation

$$D\pi(\mathbf{R})|_{\mathbf{H}_{\mathbf{R}}}^{-1}[\text{Log}_{[\mathbf{R}]}([\hat{\mathbf{R}}])] = (R_1\Omega_1, \dots, R_N\Omega_N) \in \mathbf{H}_{\mathbf{R}}, \quad (3.26)$$

the  $\Omega_i$ 's are skew-symmetric matrices solution of:

$$\min_{\Omega_i \in \mathfrak{so}(n), Q \in \text{SO}(n)} \|\Omega_1\|_F^2 + \dots + \|\Omega_N\|_F^2, \quad (3.27)$$

$$\text{such that } R_i \exp(\Omega_i) = \hat{R}_i Q, i = 1 \dots N, \text{ and} \quad (3.28)$$

$$\Omega_1 + \dots + \Omega_N = 0. \quad (3.29)$$

The rotation  $Q$  sweeps through all members of the equivalence class  $[\hat{\mathbf{R}}]$  in search of the one closest to  $\mathbf{R}$ . By substituting  $\Omega_i = \log(R_i^\top \hat{R}_i Q)$  in the objective function, we find that the objective value as a function of  $Q$  is  $\sum_{i=1}^N \|\log(R_i^\top \hat{R}_i Q)\|_F^2$ . Critical points of this function w.r.t.  $Q$  verify  $\sum_{i=1}^N \Omega_i = 0$ , hence we need not enforce the last constraint: all candidate solutions are horizontal vectors (which is not surprising). Summing up, we find that the squared Riemannian distance on  $\mathcal{P}_\emptyset$  obeys:

$$\text{dist}^2([\mathbf{R}], [\hat{\mathbf{R}}]) = \min_{Q \in \text{SO}(n)} \sum_{i=1}^N \left\| \log(R_i^\top \hat{R}_i Q) \right\|_F^2. \quad (3.30)$$

Since  $\text{SO}(n)$  is compact, this is a well-defined quantity. Let  $Q \in \text{SO}(n)$  be one of the global minimizers. Then, an acceptable value for the logarithmic map is

$$D\pi(\mathbf{R})|_{\mathbf{H}_{\mathbf{R}}}^{-1}[\text{Log}_{[\mathbf{R}]}([\hat{\mathbf{R}}])] = \left( R_1 \log(R_1^\top \hat{R}_1 Q), \dots, R_N \log(R_N^\top \hat{R}_N Q) \right). \quad (3.31)$$

Under reasonable proximity conditions on  $[\mathbf{R}]$  and  $[\hat{\mathbf{R}}]$ , the global maximizer  $Q$  is uniquely defined, and hence so is the logarithmic map. Appendix A details uniqueness conditions and algorithms to compute  $Q$ , in connection with the literature about averaging on  $\text{SO}(n)$ . For the sake of completeness, the same appendix describes embedded distances too.

## 4 Measures, integrals and distributions on $\text{SO}(n)$

The special orthogonal group  $\text{SO}(n)$ , being a compact Lie group, admits a unique bi-invariant Haar measure  $\mu$  such that  $\mu(\text{SO}(n)) = 1$  [6, Thm 3.6, p. 247]. Such a measure verifies, for all measurable subsets  $S \subset \text{SO}(n)$  and for all  $L, R \in \text{SO}(n)$ , that  $\mu(LSR) = \mu(S)$ , where  $LSR \triangleq \{LQR : Q \in S\} \subset \text{SO}(n)$ , that is, the measure of a portion of  $\text{SO}(n)$  is invariant under left and right actions of  $\text{SO}(n)$ . We will need something slightly more general:

**Lemma 4.1** (extended bi-invariance).  *$\forall L, R \in \text{O}(n)$  such that  $\det(LR) = 1$ ,  $\forall S \subset \text{SO}(n)$  measurable,  $\mu(LSR) = \mu(S)$  holds.*

*Proof.*  $LSR$  is still a measurable subset of  $SO(n)$ . Let  $\mu'$  denote the Haar measure on  $O(n) \supset SO(n)$ . The restriction of  $\mu'$  to the measurables of  $SO(n)$  is still a Haar measure. By the uniqueness of the Haar measure up to multiplicative constant, there exists  $\alpha > 0$  such that for all measurable subsets  $T \subset SO(n)$ , we have  $\mu(T) = \alpha\mu'(T)$ . Then,  $\mu(LSR) = \alpha\mu'(LSR) = \alpha\mu'(S) = \mu(S)$ .  $\square$

From the general theory of Lebesgue integration, we get a notion of integrals over  $SO(n)$  associated to the measure  $\mu$ . Lemma 4.1 then translates into the following property, with  $f : SO(n) \rightarrow \mathbb{R}$  an integrable function:

$$\forall L, R \in O(n) \text{ s.t. } \det(LR) = 1, \quad \int_{SO(n)} f(LZR) d\mu(Z) = \int_{SO(n)} f(Z) d\mu(Z). \quad (4.1)$$

This property will play an important role in the sequel.

Functions  $f$  that are invariant under conjugation, meaning that for all  $Z, Q \in SO(n)$  we have  $f(Z) = f(QZQ^{-1})$ , are *class functions*. When the integrand in  $\int_{SO(n)} f(Z) d\mu(Z)$  is a class function, we are in a position to use the Weyl integration formula specialized to  $SO(n)$  [9, Exercise 18.1–2]. All spectral functions (Assumption 3) are class functions (the converse is also true for  $SO(2k+1)$  but not for  $SO(2k)$ ). Weyl's formula comes in two flavors depending on the parity of  $n$ , and essentially reduces integrals on  $SO(n)$  to integrals over tori of dimension  $\lfloor n/2 \rfloor$ . In particular, for  $n = 2$  or  $3$ , Weyl's formula for class functions  $f$  reads:

$$\begin{aligned} \int_{SO(2)} f(Z) d\mu(Z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} d\theta, \\ \int_{SO(3)} f(Z) d\mu(Z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (1 - \cos \theta) d\theta, \end{aligned} \quad (4.2)$$

For  $n = 4$ , Weyl's formula is a double integral:

$$\begin{aligned} \int_{SO(4)} f(Z) d\mu(Z) &= \frac{1}{4(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \left( \text{diag} \left( \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}, \begin{pmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{pmatrix} \right) \right) \\ &\quad \times |e^{i\theta_1} - e^{i\theta_2}|^2 \cdot |e^{i\theta_1} - e^{-i\theta_2}|^2 d\theta_1 d\theta_2. \end{aligned} \quad (4.3)$$

Once  $SO(n)$  is equipped with a measure  $\mu$  and accompanying integral notion, we can define distributions of random variables on  $SO(n)$  via probability density functions (pdf). In general, a pdf on  $SO(n)$  is a nonnegative measurable function  $f$  on  $SO(n)$  such that  $\int_{SO(n)} f(Z) d\mu(Z) = 1$ . In this work, for convenience, we further assume pdf's are smooth and positive (Assumption 1), as we will need to compute the derivatives of their logarithm.

**Example 1** (uniform). *The pdf associated to the uniform distribution is  $f(Z) \equiv 1$ , since we normalized the Haar measure such that  $\mu(SO(n)) = 1$ . We write  $Z \sim \text{Uni}(SO(n))$  to mean that  $Z$  is a uniformly distributed random rotation. A number of algorithms exist to generate pseudo-random rotation matrices from the uniform distribution [11, §2.5.1] [15]. Possibly one of the easiest methods to implement is the following  $\mathcal{O}(n^3)$  algorithm, adapted from [15, Method A, p. 22] with implementation details as in [26] (for large  $n$ , see the former paper for algorithms with better complexity):*

1. Generate  $A \in \mathbb{R}^{n \times n}$ , such that the entries  $A_{ij} \sim \mathcal{N}(0, 1)$  are i.i.d. normal random variables;
2. Obtain a QR decomposition of  $A$ :  $QR = A$ ;
3. Set  $Q := Q \text{diag}(\text{sign}(\text{diag}(R)))$  (this ensures the mapping  $A \mapsto Q$  is well-defined; see [26]);
4.  $Q$  is now uniform on  $O(n)$ . If  $\det(Q) = -1$ , permute columns 1 and 2 of  $Q$ . Return  $Q$ .

**Example 2** (isotropic Langevin). *The isotropic Langevin distribution on  $SO(n)$  with mean  $Q \in SO(n)$  and concentration  $\kappa \geq 0$  has pdf*

$$f(Z) = \frac{1}{c_n(\kappa)} \exp(\kappa \text{trace}(Q^\top Z)), \quad (4.4)$$

where  $c_n(\kappa)$  is a normalization constant such that  $f$  has unit mass. We write  $Z \sim \text{Lang}(Q, \kappa)$  to mean that  $Z$  is a random variable with pdf (4.4). For  $\kappa = 0$ ,  $Z \sim \text{Uni}(\text{SO}(n))$ ; in the limit  $\kappa \rightarrow \infty$ ,  $Z = Q$  w.p. 1. The isotropic Langevin distribution has a Gaussian-like shape. The Langevin pdf centered around  $Q = I$  is a spectral function, i.e., it fulfills Assumption 3.

The larger the concentration parameter  $\kappa$ , the more the distribution is concentrated around the mean  $Q$ . By bi-invariance of  $\mu$ ,  $c_n(\kappa)$  is independent of  $Q$ :

$$c_n(\kappa) = \int_{\text{SO}(n)} \exp(\kappa \text{trace}(Q^\top Z)) d\mu(Z) = \int_{\text{SO}(n)} \exp(\kappa \text{trace}(Z)) d\mu(Z). \quad (4.5)$$

Since the integrand is a class function, Weyl's integration formulas apply for any value of  $n$ . Using (4.2) and (4.3), we work out explicit formulas for  $c_n(\kappa)$ ,  $n = 2, 3, 4$ :

$$c_2(\kappa) = I_0(2\kappa), \quad (4.6)$$

$$c_3(\kappa) = \exp(\kappa)(I_0(2\kappa) - I_1(2\kappa)), \quad (4.7)$$

$$c_4(\kappa) = I_0(2\kappa)^2 - 2I_1(2\kappa)^2 + I_0(2\kappa)I_2(2\kappa), \quad (4.8)$$

in terms of the modified Bessel functions of the first kind,  $I_\nu$  [39]. See Appendix B for details.

For  $n = 2$ , the Langevin distribution is also known as the von Mises or Fisher distribution on the circle [25]. The Langevin distribution on  $\text{SO}(n)$  also exists in anisotropic form [12]. Unfortunately, the associated pdf is no longer a spectral function even for  $Q = I$ , which is an instrumental property in the present work. Consequently, we do not treat anisotropic distributions for now. Chikuse gives an in-depth treatment of statistics on the Grassmann and Stiefel manifolds [11], including a study of Langevin distributions on  $\text{SO}(n)$  as a special case.

Based on a uniform sampling algorithm on  $\text{SO}(n)$ , it is easy to devise an acceptance-rejection scheme to sample from the Langevin distribution [11, §2.5.2]. Not surprisingly, for large values of  $\kappa$ , this tends to be very inefficient. Chiuso et al. report using a Metropolis-Hastings-type algorithm instead [12, §7]. Hoff describes an efficient Gibbs sampling method to sample from a more general family of distributions on the Stiefel manifold, which can be modified to work on  $\text{SO}(n)$  [19].

For  $\text{Lang}(I, \kappa)$ , the following formulas will prove useful:

$$\log f(Z) = \kappa \text{trace}(Z) - \log c_n(\kappa), \quad (4.9)$$

$$\nabla \log f(Z) = \kappa I_n, \quad (4.10)$$

$$\text{grad} \log f(Z) = P_Z(\nabla \log f(Z)) = -\kappa Z \text{sk}(Z), \quad (4.11)$$

where  $\nabla$  denotes the usual gradient in the embedding space  $\mathbb{R}^{n \times n}$ ,  $\text{grad}$  denotes the gradient on  $\text{SO}(n)$  and  $P_Z$  (3.4) is the orthogonal projector from the embedding space onto the tangent space  $T_Z \text{SO}(n)$ , with  $\text{sk}(Z) \triangleq (Z - Z^\top)/2$  the skew-symmetric part of  $Z$ —see Section 3.

The set of pdf's is closed under convex combinations, as is the set of functions satisfying our assumptions 1 and 3. We could therefore consider mixtures of Langevin distributions around  $I$  with various concentrations. In the next example, we combine the Langevin distribution and the uniform distribution. To the best of our knowledge, this is the first description of a heavy-tail-type distribution on  $\text{SO}(n)$ . Such a distribution may prove useful for any application involving outliers in rotation estimation.

**Example 3** (isotropic Langevin with outliers). *We define the isotropic Langevin distribution with outliers on  $\text{SO}(n)$  with mean  $Q \in \text{SO}(n)$ , concentration  $\kappa > 0$  and outlier probability  $1 - p \in [0, 1]$  via the pdf*

$$f(Z) = \frac{p}{c_n(\kappa)} \exp(\kappa \text{trace}(Q^\top Z)) + (1 - p). \quad (4.12)$$

For  $Z$  distributed as such, we write  $Z \sim \text{LangUni}(Q, \kappa, p)$ . For  $p = 1$ , this is the isotropic Langevin distribution. For  $p = 0$ , this is the uniform distribution. Notice that  $f$  is a spectral function for  $Q = I$ .

A random rotation sampled from  $\text{LangUni}(Q, \kappa, p)$  is, with probability  $p$ , sampled from  $\text{Lang}(Q, \kappa)$ , and with probability  $1 - p$ , sampled uniformly at random. Measurements of  $Q$  distributed in such a

way are outliers with probability  $1 - p$ , i.e., bear no information about  $Q$  w.p.  $1 - p$ . For  $0 < p < 1$ , the LangUni is a kind of heavy-tailed distribution on  $\text{SO}(n)$ .

For  $\text{LangUni}(I, \kappa, p)$ , the following will prove useful:

$$\text{grad} \log f(Z) = -\frac{p\kappa \exp(\kappa \text{trace } Z)}{c_n(\kappa)f(Z)} Z \text{sk}(Z), \quad (4.13)$$

with  $\text{sk}(Z) = (Z - Z^\top)/2$  denoting the skew-symmetric part of  $Z$ .

To conclude this section, we remark more broadly that all isotropic distributions around the identity matrix have a spectral pdf. Indeed, let  $f : \text{SO}(n) \rightarrow \mathbb{R}$  be isotropic w.r.t. the geodesic distance on  $\text{SO}(n)$ ,  $\text{dist}(R_1, R_2) = \|\log(R_1^\top R_2)\|_F$  (3.6). That is, there is a function  $\tilde{f}$  such that  $f(Z) = \tilde{f}(\text{dist}(I, Z)) = \tilde{f}(\|\log Z\|_F)$ . It is then obvious that  $f(QZQ^\top) = f(Z)$  for all  $Q \in \text{O}(n)$  since  $\log(QZQ^\top) = Q \log(Z) Q^\top$ . The same holds for the embedded distance  $\text{dist}(R_1, R_2) = \|R_1 - R_2\|_F$ . This shows that the assumptions proposed in Section 2 include many interesting distributions.

Similarly we establish that all spectral pdf's have zero bias around the identity matrix  $I$ . The bias is the tangent vector (skew-symmetric matrix)  $\Omega = \mathbb{E}\{\text{Log}_I(Z)\}$ , with  $Z \sim f$ ,  $f$  spectral. Since  $\text{Log}_I(Z) = \log(Z)$  (3.5), we find, with a change of variable  $Z := QZQ^\top$  going from the first to the second integral, that for all  $Q \in \text{O}(n)$ :

$$\Omega = \int_{\text{SO}(n)} \log(Z) f(Z) d\mu(Z) = \int_{\text{SO}(n)} \log(QZQ^\top) f(Z) d\mu(Z) = Q\Omega Q^\top. \quad (4.14)$$

Indeed,  $\log(QZQ^\top) = Q \log(Z) Q^\top$ . Since skew-symmetric matrices are normal matrices and since  $\Omega$  and  $\Omega^\top = -\Omega$  have the same eigenvalues, we may choose  $Q \in \text{O}(n)$  such that  $Q\Omega Q^\top = -\Omega$ . Therefore,  $\Omega = -\Omega = 0$ . As a consequence, it is only possible to treat unbiased measurements under the assumptions we make in this paper.

## 5 The Fisher information metric for synchronization

As described in Section 2, the relative rotation measurements  $H_{ij} = Z_{ij} R_i R_j^\top$  (2.3) reveal information about the true (but unknown) rotations  $R_1, \dots, R_N$ . The Fisher information matrix (FIM) we compute here encodes how much information these measurements contain on average. In other words, the FIM is an assessment of the quality of the measurements we have at our disposal for the purpose of estimating the sought parameters. The FIM will be instrumental in deriving Cramér-Rao bounds in the next section.

The FIM is a standard object of study for estimation problems on Euclidean spaces. In the setting of synchronization of rotations, the parameter space is a manifold and we thus need a more general definition of the FIM. We quote, mutatis mutandis, the definition of FIM as stated in [7] following [36]:

**Definition 1** (FIM). *Let  $\mathcal{P}$  be the parameter space of an estimation problem and let  $\theta \in \mathcal{P}$  be the (unknown) parameter. Let  $f(y; \theta)$  be the pdf of the measurement  $y$  conditioned by  $\theta$ . The log-likelihood function  $L : \mathcal{P} \rightarrow \mathbb{R}$  is  $L(\theta) = \log f(y; \theta)$ . Let  $e = \{e_1, \dots, e_d\}$  be an orthonormal basis of the tangent space  $T_\theta \mathcal{P}$  w.r.t. the Riemannian metric  $\langle \cdot, \cdot \rangle_\theta$ . The Fisher information matrix of the estimation problem on  $\mathcal{P}$  w.r.t. the basis  $e$  is defined by:*

$$F_{ij} = \mathbb{E} \left\{ \langle \text{grad } L(\theta), e_i \rangle_\theta \cdot \langle \text{grad } L(\theta), e_j \rangle_\theta \right\}. \quad (5.1)$$

*Expectation is taken w.r.t. the measurement  $y$ .*

Hence, we need the gradient of the log-likelihood function  $L$  evaluated at the true rotations  $\mathbf{R} \in \mathcal{P}$ . The  $i^{\text{th}}$  component of that gradient is equal to (see (3.13)):

$$\text{grad}_i L(\mathbf{R}) = \sum_{j \in V_i} [\text{grad} \log f_{ij}(Z_{ij})]^\top Z_{ij} R_i. \quad (5.2)$$

The vector field  $\text{grad log } f_{ij}$  on  $\text{SO}(n)$  may be factored into:

$$\text{grad log } f_{ij}(Z) = ZG_{ij}^\top(Z), \quad (5.3)$$

where  $G_{ij} : \text{SO}(n) \mapsto \mathfrak{so}(n)$  is a mapping that will play an important role in the sequel. In particular, the  $i^{\text{th}}$  gradient component now takes the short form:

$$\text{grad}_i L(\mathbf{R}) = \sum_{j \in V_i} G_{ij}(Z_{ij})R_i. \quad (5.4)$$

Let us consider a canonical orthonormal basis of  $\mathfrak{so}(n)$ :  $(E_1, \dots, E_d)$ , where  $d = \dim \mathfrak{so}(n) = \binom{n}{2}$ . For  $n = 3$ , we pick this one:

$$E_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (5.5)$$

An obvious generalization yields similar bases for other values of  $n$ . We can transport this canonical basis into an orthonormal basis for the tangent space  $T_{R_i}\text{SO}(n)$  as  $(R_i E_1, \dots, R_i E_d)$ . To compute the Fisher information matrix, we need an orthonormal basis for the tangent space  $T_{\mathbf{R}}\mathcal{P}$  to the manifold  $\mathcal{P} = \text{SO}(n) \times \dots \times \text{SO}(n)$ . This matrix, of dimension  $dN \times dN$ , is composed of  $N \times N$  blocs of size  $d \times d$ . Let us index the  $(k, \ell)$  entry inside the  $(i, j)$  bloc as  $F_{ij, k\ell}$ . Let us also fix an orthonormal basis for the tangent space at  $\mathbf{R} = (R_1, \dots, R_N)$  of  $\mathcal{P}$ , as

$$(\xi_{ik})_{i=1\dots N, k=1\dots d}, \text{ with } \xi_{ik} = (0, \dots, 0, R_i E_k, 0, \dots, 0),$$

a zero vector except for the  $i^{\text{th}}$  component equal to  $R_i E_k$ . (5.6)

Accordingly, the matrix  $F$  at  $\mathbf{R}$  is defined by (see Definition 1):

$$\begin{aligned} F_{ij, k\ell} &= \mathbb{E} \{ \langle \text{grad } L(\mathbf{R}), \xi_{ik} \rangle \cdot \langle \text{grad } L(\mathbf{R}), \xi_{j\ell} \rangle \} \\ &= \mathbb{E} \{ \langle \text{grad}_i L(\mathbf{R}), R_i E_k \rangle \cdot \langle \text{grad}_j L(\mathbf{R}), R_j E_\ell \rangle \} \\ &= \sum_{r \in V_i} \sum_{s \in V_j} \mathbb{E} \{ \langle G_{ir}(Z_{ir}), R_i E_k R_i^\top \rangle \cdot \langle G_{js}(Z_{js}), R_j E_\ell R_j^\top \rangle \}. \end{aligned} \quad (5.7)$$

We prove that, in expectation, the mappings  $G_{ij} : \text{SO}(n) \rightarrow \mathfrak{so}(n)$  are zero. This fact is directly related to the standard result from estimation theory stating that the average score  $V(\theta) = \mathbb{E} \{ \text{grad log } f(y; \theta) \}$  for a given parameterized probability density function  $f$  is zero.

**Lemma 5.1.** *Given a probability density function  $f : \text{SO}(n) \rightarrow \mathbb{R}^+$  and the mapping  $G : \text{SO}(n) \rightarrow \mathfrak{so}(n)$  such that  $\text{grad log } f(Z) = ZG(Z)$ , we have that  $\mathbb{E} \{ G(Z) \} = 0$ , where expectation is taken w.r.t.  $Z$ , distributed according to  $f$ .*

*Proof.* Define  $h(Q) = \int_{\text{SO}(n)} f(ZQ) d\mu(Z)$  for  $Q \in \text{SO}(n)$ . Since  $f$  is a probability density function, bi-invariance of  $\mu$  (4.1) yields  $h(Q) \equiv 1$ . Taking gradients with respect to the parameter  $Q$ , we get:

$$0 = \text{grad } h(Q) = \int_{\text{SO}(n)} \text{grad}_Q f(ZQ) d\mu(Z) = \int_{\text{SO}(n)} Z^\top \text{grad } f(ZQ) d\mu(Z).$$

With a change of variable  $Z := ZQ$ , by bi-invariance of  $\mu$ , we further obtain:

$$\int_{\text{SO}(n)} Z^\top \text{grad } f(Z) d\mu(Z) = 0.$$

Using this last result and the fact that  $\text{grad log } f(Z) = \frac{1}{f(Z)} \text{grad } f(Z)$ , we conclude:

$$\mathbb{E} \{ G(Z) \} = \int_{\text{SO}(n)} Z^\top \text{grad log } f(Z) f(Z) d\mu(Z) = \int_{\text{SO}(n)} Z^\top \text{grad } f(Z) d\mu(Z) = 0. \quad \square$$

We now invoke Assumption 2 (independence). Independence of  $Z_{ij}$  and  $Z_{pq}$  for two distinct edges  $(i, j)$  and  $(p, q)$  implies that, for any two functions  $\phi_1, \phi_2 : \text{SO}(n) \rightarrow \mathbb{R}$ , it holds that

$$\mathbb{E} \{ \phi_1(Z_{ij}) \phi_2(Z_{pq}) \} = \mathbb{E} \{ \phi_1(Z_{ij}) \} \mathbb{E} \{ \phi_2(Z_{pq}) \},$$

provided all involved expectations exist. Using both this and Lemma 5.1, most terms in (5.7) vanish and we obtain a simplified expression for the matrix  $F$ :

$$F_{ij,kl} = \begin{cases} \sum_{r \in V_i} \mathbb{E} \{ \langle G_{ir}(Z_{ir}), R_i E_k R_i^\top \rangle \cdot \langle G_{ir}(Z_{ir}), R_i E_\ell R_i^\top \rangle \}, & \text{if } i = j, \\ \mathbb{E} \{ \langle G_{ij}(Z_{ij}), R_i E_k R_i^\top \rangle \cdot \langle G_{ji}(Z_{ji}), R_j E_\ell R_j^\top \rangle \}, & \text{if } i \neq j \text{ and } (i, j) \in \mathcal{E}, \\ 0, & \text{if } i \neq j \text{ and } (i, j) \notin \mathcal{E}. \end{cases} \quad (5.8)$$

We further manipulate the second case, which involves both  $G_{ij}$  and  $G_{ji}$ , by noting that those are deterministically linked. Indeed, by symmetry of the measurements ( $H_{ij} = H_{ji}^\top$ ), we have that (i)  $Z_{ji} = R_j R_i^\top Z_{ij}^\top R_i R_j^\top$  and (ii)  $f_{ij}(Z_{ij}) = f_{ji}(Z_{ji})$ . Invoking Assumption 3, since  $Z_{ij}$  and  $Z_{ji}$  have the same eigenvalues, it follows that  $f_{ij}(Z) = f_{ji}(Z)$  for all  $Z \in \text{SO}(n)$ . As a by-product, it also holds that  $G_{ij}(Z) = G_{ji}(Z)$  for all  $Z \in \text{SO}(n)$ . Still under Assumption 3, we show in Appendix D that

$$\forall Q \in \text{O}(n), \quad G_{ij}(Q Z Q^\top) = Q G_{ij}(Z) Q^\top, \quad \text{and} \quad G_{ij}(Z^\top) = -G_{ij}(Z). \quad (5.9)$$

Combining these observations, we obtain:

$$G_{ji}(Z_{ji}) = G_{ij}(Z_{ji}) = G_{ij}(R_j R_i^\top Z_{ij}^\top R_i R_j^\top) = -R_j R_i^\top G_{ij}(Z_{ij}) R_i R_j^\top. \quad (5.10)$$

The minus sign, which plays an important role in the structure of the FIM, comes about via the skew-symmetry of  $G_{ij}$ . The following identity thus holds:

$$\langle G_{ji}(Z_{ji}), R_j E_\ell R_j^\top \rangle = -\langle G_{ij}(Z_{ij}), R_i E_\ell R_i^\top \rangle. \quad (5.11)$$

This can advantageously be plugged into (5.8).

We set out to describe the expectations appearing in (5.8), which will take us through a couple of lemmas. Let us, for a certain pair  $(i, j) \in \mathcal{E}$ , introduce the functions  $h_k : \text{SO}(n) \rightarrow \mathbb{R}$ ,  $k = 1 \dots d$ :

$$h_k(Z) \triangleq \langle G_{ij}(Z), R_i E_k R_i^\top \rangle, \quad (5.12)$$

where we chose to not overload the notation  $h_k$  with an explicit reference to the pair  $(i, j)$ , as this will always be clear from the context. We may rewrite the FIM in terms of the functions  $h_k$ , starting from (5.8) and incorporating (5.11):

$$F_{ij,kl} = \begin{cases} \sum_{r \in V_i} \mathbb{E} \{ h_k(Z_{ir}) \cdot h_\ell(Z_{ir}) \}, & \text{if } i = j, \\ -\mathbb{E} \{ h_k(Z_{ij}) \cdot h_\ell(Z_{ij}) \}, & \text{if } i \neq j \text{ and } (i, j) \in \mathcal{E}, \\ 0, & \text{if } i \neq j \text{ and } (i, j) \notin \mathcal{E}. \end{cases} \quad (5.13)$$

Another consequence of Assumption 3 is that the functions  $h_k(Z)$  and  $h_\ell(Z)$  are uncorrelated for  $k \neq \ell$ , where  $Z$  is distributed according to the density  $f_{ij}$ . As a consequence,  $F_{ij,kl} = 0$  for  $k \neq \ell$ , i.e., the  $d \times d$  blocks of  $F$  are diagonal. We establish this fact in Lemma 5.3, right after a technical lemma.

**Lemma 5.2.** *Let  $E, E' \in \mathfrak{so}(n)$  such that  $E_{ij} = -E_{ji} = 1$  and  $E'_{kl} = -E'_{lk} = 1$  (all other entries are zero), with  $\langle E, E' \rangle = 0$ , i.e.,  $\{i, j\} \neq \{k, \ell\}$ . Then, there exists  $P \in \text{O}(n)$  a signed permutation such that  $P^\top E P = E'$  and  $P^\top E' P = -E$ .*

*Proof.* See Appendix E. □



**Lemma 5.3.** *Let  $Z \in \text{SO}(n)$  be a random variable distributed according to  $f_{ij}$ . The random variables  $h_k(Z)$  and  $h_\ell(Z)$ ,  $k \neq \ell$ , as defined in (5.12) have zero mean and are uncorrelated, i.e.,  $\mathbb{E}\{h_k(Z)\} = \mathbb{E}\{h_\ell(Z)\} = 0$  and  $\mathbb{E}\{h_k(Z) \cdot h_\ell(Z)\} = 0$ . Furthermore, it holds that  $\mathbb{E}\{h_k^2(Z)\} = \mathbb{E}\{h_\ell^2(Z)\}$ .*

*Proof.* The first part follows directly from Lemma 5.1. We show the second part. Consider a signed permutation matrix  $P_{k\ell} \in \text{O}(n)$  such that  $P_{k\ell}^\top E_k P_{k\ell} = E_\ell$  and  $P_{k\ell}^\top E_\ell P_{k\ell} = -E_k$ . Such a matrix always exists according to Lemma 5.2. Then, identity (5.9) yields:

$$h_k(R_i P_{k\ell} R_i^\top Z R_i P_{k\ell}^\top R_i^\top) = \langle G_{ij}(Z), R_i P_{k\ell}^\top E_k P_{k\ell} R_i^\top \rangle = h_\ell(Z). \quad (5.14)$$

Likewise,

$$h_\ell(R_i P_{k\ell} R_i^\top Z R_i P_{k\ell}^\top R_i^\top) = -h_k(Z). \quad (5.15)$$

These identities as well as the (extended) bi-invariance (4.1) of the Haar measure  $\mu$  on  $\text{SO}(n)$  and the fact that  $f_{ij}$  is a spectral function yield, using the change of variable  $Z := R_i P_{k\ell} R_i^\top Z R_i P_{k\ell}^\top R_i^\top$  going from the first to the second integral:

$$\mathbb{E}\{h_k(Z) \cdot h_\ell(Z)\} = \int_{\text{SO}(n)} h_k(Z) h_\ell(Z) f_{ij}(Z) d\mu(Z) \quad (5.16)$$

$$= \int_{\text{SO}(n)} -h_\ell(Z) h_k(Z) f_{ij}(Z) d\mu(Z) = -\mathbb{E}\{h_k(Z) \cdot h_\ell(Z)\}. \quad (5.17)$$

Hence,  $\mathbb{E}\{h_k(Z) \cdot h_\ell(Z)\} = 0$ . We prove the last statement using the same change of variable:

$$\mathbb{E}\{h_k^2(Z)\} = \int_{\text{SO}(n)} h_k^2(Z) f_{ij}(Z) d\mu(Z) = \int_{\text{SO}(n)} h_\ell^2(Z) f_{ij}(Z) d\mu(Z) = \mathbb{E}\{h_\ell^2(Z)\}.$$

We note that, more generally, it can be shown that the  $h_k$ 's are identically distributed.  $\square$

The skew-symmetric matrices  $(R_i E_1 R_i^\top, \dots, R_i E_d R_i^\top)$ ,  $d = \dim \mathfrak{so}(n)$ , form an orthonormal basis of the Lie algebra  $\mathfrak{so}(n)$ . Consequently, we may expand each mapping  $G_{ij}$  in this basis and express its squared norm as:

$$G_{ij}(Z) = \sum_{k=1}^d h_k(Z) \cdot R_i E_k R_i^\top, \quad \|G_{ij}(Z)\|^2 = \sum_{k=1}^d h_k^2(Z). \quad (5.18)$$

Since by Lemma 5.3 the quantity  $\mathbb{E}\{h_k^2(Z_{ij})\}$  does not depend on  $k$ , it follows that:

$$\mathbb{E}\{h_k^2(Z_{ij})\} = \frac{1}{d} \mathbb{E}\{\|G_{ij}(Z_{ij})\|^2\}, \quad k = 1 \dots d. \quad (5.19)$$

This further shows that the  $d \times d$  blocks that constitute the FIM have constant diagonal. Hence,  $F$  can be expressed as the Kronecker product ( $\otimes$ ) of some matrix with the identity  $I_d$ . Let us define the following (positive) weights on the edges of the measurement graph:

$$w_{ij} = w_{ji} = \mathbb{E}\{\|G_{ij}(Z_{ij})\|^2\} \triangleq \mathbb{E}\{\|\text{grad log } f_{ij}(Z_{ij})\|^2\}, \quad \forall (i, j) \in \mathcal{E}. \quad (5.20)$$

Then, the *weighted Laplacian matrix*  $\mathcal{L} \in \mathbb{R}^{N \times N}$ ,  $\mathcal{L} = \mathcal{L}^\top \succeq 0$ , of the weighted measurement graph is given by:

$$\mathcal{L}_{ij} = \begin{cases} \sum_{r \in V_i} w_{ir}, & \text{if } i = j, \\ -w_{ij}, & \text{if } i \neq j \text{ and } (i, j) \in \mathcal{E}, \\ 0, & \text{if } i \neq j \text{ and } (i, j) \notin \mathcal{E}. \end{cases} \quad (5.21)$$

It is now apparent that the matrix  $F \in \mathbb{R}^{dN \times dN}$ —see equation (5.13)—is tightly related to the weighted Laplacian matrix  $\mathcal{L}$  (5.21). We summarize this in the following theorem.

**Theorem 5.4** (FIM for synchronization). *Let  $R_1, \dots, R_N \in \text{SO}(n)$  be unknown but fixed rotations and let  $H_{ij} = Z_{ij} R_i R_j^\top$  for  $(i, j) \in \mathcal{E}, i < j$ , with the  $Z_{ij}$ 's random rotations whose distributions fulfill Assumptions 1–3. We consider the problem of estimating the  $R_i$ 's given a realization of the  $H_{ij}$ 's. The Fisher information matrix (Definition 1) of that estimation problem with respect to the basis (5.6) is given by*

$$F = \frac{1}{d}(\mathcal{L} \otimes I_d), \quad (5.22)$$

where  $\otimes$  denotes the Kronecker product,  $d = \dim \text{SO}(n) = n(n-1)/2$ ,  $I_d$  is the  $d \times d$  identity matrix and  $\mathcal{L}$  is the weighted Laplacian matrix of the measurement graph defined by (5.21).

The Laplacian matrix has a number of properties, some of which will yield nice interpretations when deriving the Cramér-Rao bounds. One remarkable fact is that the Fisher information matrix does not depend on  $\mathbf{R} = (R_1, \dots, R_N)$ , the set of true rotations. This is an appreciable property seen as  $\mathbf{R}$  is unknown in practice. This stems from the strong symmetries in our problem.

Another important feature of this FIM is that it is rank deficient. Indeed, for a connected measurement graph,  $\mathcal{L}$  has exactly one zero eigenvalue (and more if the graph is disconnected) associated to the kernel vector of all ones,  $\mathbf{1}_N$ . The kernel of the FIM is thus composed of all vectors of the form  $\mathbf{1}_N \otimes t$ , with  $t \in \mathbb{R}^d$  arbitrary. This corresponds to the vertical spaces of  $\mathcal{P}$  w.r.t. the equivalence relation (3.17), i.e., the kernel consists in all tangent vectors that move all rotations  $R_i$  in the same direction, leaving their relative positions unaffected. This makes perfect sense: the distribution of the measurements  $H_{ij}$  is also unaffected by such changes, hence the FIM, seen as a quadratic form, takes up a zero value when applied to the corresponding vectors. We will need special tools to deal with this (structured) singularity when deriving the CRB's in the next section.

Notice how Assumption 2 (independence) gave  $F$  a bloc structure based on the sparsity pattern of the Laplacian matrix, while Assumption 3 (spectral pdf's) made each bloc proportional to the  $d \times d$  identity matrix and made  $F$  independent of  $\mathbf{R}$ .

In the two following examples, we explicitly compute the weights  $w_{ij}$  (5.20) associated to two types of noise distributions: (1) unbiased isotropic Langevin distributions (akin to Gaussian noise), and (2) a mix of the former distribution and complete outliers (uniform distribution)—see Section 4. Combining formulas for the weights  $w_{ij}$  and equations (5.21) and (5.22), one can compute the Fisher information matrix explicitly.

**Example 4** (Langevin distributions). *(Continued from Example 2) Let us consider the isotropic Langevin distribution over  $\text{SO}(n)$ , with concentration parameter  $\kappa > 0$  and mean at the identity matrix:  $f(Z) = \frac{1}{c_n(\kappa)} \exp(\kappa \text{trace} Z)$ . From the expression of  $\text{grad} \log f(Z)$  (4.11), we find that  $G(Z) = \kappa \text{sk}(Z) = \frac{\kappa}{2}(Z - Z^\top)$ . Hence, the weight  $w$  associated to the noise distribution  $f$  is a function  $\alpha_n(\kappa)$  given by:*

$$w = \alpha_n(\kappa) = \mathbb{E} \{ \|G(Z)\|^2 \} = \frac{\kappa^2}{4} \int_{\text{SO}(n)} \|Z - Z^\top\|^2 f(Z) d\mu(Z). \quad (5.23)$$

Since the integrand is again a class function, we may evaluate this integral using Weyl's integration formulas—see Section 4 and Appendix B for an example. In particular, for  $n = 2$  or  $3$ , identities (4.2) apply and we derive the following expressions:

$$\alpha_2(\kappa) = \kappa \frac{I_1(2\kappa)}{I_0(2\kappa)}, \quad \alpha_3(\kappa) = \frac{\kappa}{2} \frac{(2 - \kappa)I_1(2\kappa) + \kappa I_3(2\kappa)}{I_0(2\kappa) - I_1(2\kappa)}. \quad (5.24)$$

The functions  $I_\nu(z)$  are the modified Bessel functions of the first kind (B.8). We used the formulas for the normalization constants  $c_2(\kappa)$  (4.6) and  $c_3(\kappa)$  (4.7) as well as the identity  $I_1(2\kappa) = \kappa(I_0(2\kappa) - I_2(2\kappa))$ .

For the special case  $n = 2$ , taking the concentrations for all measurements to be equal to  $\kappa$ , we find that the Fisher information matrix  $F$  is equal to the weighted Laplacian matrix  $\mathcal{L}$ , which is proportional to the unweighted Laplacian matrix  $\mathcal{L}_0 = D - A$ , with  $D$  the degree matrix and  $A$  the adjacency matrix of the measurement graph:

$$F = \mathcal{L} = \alpha_2(\kappa) \mathcal{L}_0. \quad (5.25)$$

This particular result was shown before via another method in [21]. For the derivation in the latter work, commutativity of rotations in the plane is instrumental, and hence the proof method does not—at least in the proposed form—transfer to  $\text{SO}(n)$  for  $n \geq 3$ .

**Example 5** (Langevin/outlier distributions). *(Continued from Example 3)* Given a concentration  $\kappa > 0$  and a probability  $p \in [0, 1]$ , the p.d.f.

$$f(Z) = \frac{p}{c_n(\kappa)} \exp(\kappa \text{trace } Z) + 1 - p \quad (5.26)$$

corresponds to picking a rotation, with probability  $p$ , around the identity matrix following an isotropic Langevin distribution with concentration  $\kappa$ , or, with probability  $1 - p$ , uniformly at random anywhere on  $\text{SO}(n)$ . This pdf satisfies all our assumptions. The formula for  $\text{grad log } f(Z)$  (4.13) yields  $G(Z) = \frac{1}{f(Z)} \frac{p\kappa \exp(\kappa \text{trace } Z)}{c_n(\kappa)} \text{sk}(Z)$ . We thus need to evaluate

$$w = \alpha_n(\kappa, p) = \int_{\text{SO}(n)} \frac{1}{f(Z)} \left( \frac{p\kappa \exp(\kappa \text{trace } Z)}{2c_n(\kappa)} \right)^2 \|Z - Z^\top\|^2 d\mu(Z). \quad (5.27)$$

The integrand being a class function again, Weyl’s formula applies and after some algebra using the material in Section 4, we obtain formulas  $\alpha_n(\kappa, p)$  for  $n = 2$  or  $3$ :

$$\alpha_2(\kappa, p) = \frac{(p\kappa)^2}{c_2(\kappa)} \frac{1}{\pi} \int_0^\pi \frac{(1 - \cos 2\theta) \exp(4\kappa \cos \theta)}{p \exp(2\kappa \cos \theta) + (1 - p)c_2(\kappa)} d\theta, \quad (5.28)$$

$$\alpha_3(\kappa, p) = \frac{(p\kappa)^2 \exp(2\kappa)}{c_3(\kappa)} \frac{1}{\pi} \int_0^\pi \frac{(1 - \cos 2\theta)(1 - \cos \theta) \exp(4\kappa \cos \theta)}{p \exp(\kappa(1 + 2 \cos \theta)) + (1 - p)c_3(\kappa)} d\theta. \quad (5.29)$$

These integrals may be evaluated numerically. The same machinery goes through for  $n \geq 4$ .

## 6 Cramér-Rao bounds for synchronization

Classical Cramér-Rao bounds (CRB’s) give a lower bound on the covariance matrix  $C$  of any unbiased estimator for an estimation problem in  $\mathbb{R}^n$ . In terms of the Fisher information matrix  $F$  of that problem, the classical result reads  $C \succeq F^{-1}$ . In our setting, an estimation problem on a manifold with singular  $F$ , we need to resort to a more general statement of the CRB.

First off, because our parent parameter space  $\mathcal{P}$  is a manifold instead of a Euclidean space, we need generalized definitions of bias and covariance of estimators—we will quote them momentarily. And because manifolds are usually curved—as opposed to Euclidean spaces which are flat—the CRB takes up the form  $C \succeq F^{-1} + \text{curvature terms}$  [36]. We will compute the curvature terms and show that they become negligible at large signal-to-noise ratios (SNR).

Secondly, owing to the indeterminacies in our estimation problem (based on the  $H_{ij}$ ’s only, we can only recover the  $R_i$ ’s up to a global rotation), the FIM (5.22) is singular, with a kernel that is identifiable with the vertical space (3.23). In [7], CRB’s are provided for this scenario by looking at the estimation problem either on the submanifold  $\mathcal{P}_A$  (where indeterminacies have been resolved by fixing anchors) or on the quotient space  $\mathcal{P}_\emptyset$  (where each equivalence class of rotations is regarded as one point).

We should bear in mind that intrinsic Cramér-Rao bounds are fundamentally asymptotic bounds for large SNR [36]. At low SNR, the bounds may fail to capture features of the estimation problem that become dominant for large errors. In particular, since our parameter spaces  $\mathcal{P}_A$  and  $\mathcal{P}_\emptyset$  are compact, there is an upperbound on how badly one can estimate the true rotations. Because of their local nature (CRB’s focus on small error scenarios), the bounds we establish here are unable to capture this essential feature and should not be used in low SNR regimes.

We now give a definition of unbiased estimator. After this, we differentiate between the anchored and the anchor-free scenarios to establish CRB’s.

**Definition 2** (unbiased estimator). *Let  $\mathcal{P}$  be the parameter space of an estimation problem. Let  $f(y; \theta)$  be the pdf of the measurement  $y \in \mathcal{M}$  conditioned by  $\theta \in \mathcal{P}$ . An estimator is a mapping*

$\hat{\theta} : \mathcal{M} \rightarrow \mathcal{P}$  assigning a parameter  $\hat{\theta}(y)$  to every possible realization of the measurement  $y$ . The bias of an estimator is the vector field  $b$  on  $\mathcal{P}$ :

$$\forall \theta \in \mathcal{P}, \quad b(\theta) \triangleq \mathbb{E} \left\{ \text{Log}_{\theta}(\hat{\theta}(y)) \right\}. \quad (6.1)$$

An unbiased estimator has a vanishing bias  $b \equiv 0$ .

## 6.1 Anchored synchronization

When anchors are provided, the rotation matrices  $R_i$  for  $i \in A$ ,  $A \neq \emptyset$  are known. The parameter space then becomes  $\mathcal{P}_A$  (3.14), which is a Riemannian submanifold of  $\mathcal{P}$ . The synchronization problem is well-posed on  $\mathcal{P}_A$ , provided there is at least one anchor in each connected component of the measurement graph.

We set out to use [7, Theorem 4], which states Cramér-Rao bounds for estimation problems on Riemannian submanifolds. To this end, we first define the  $dN \times dN$  covariance matrix  $C_A$  for anchored synchronization, then we introduce a projected FIM  $F_A$ , and we finally state the bound.

**Definition 3** (anchored covariance). *Following [7, eq. (2)], the covariance matrix of an estimator  $\hat{\mathbf{R}}$  mapping each possible set of measurements  $H_{ij}$  to a point in  $\mathcal{P}_A$ , expressed w.r.t. the orthonormal basis (5.6) of  $\text{T}_{\mathbf{R}}\mathcal{P}$ , is given by:*

$$(C_A)_{ij,kl} = \mathbb{E} \left\{ \left\langle \text{Log}_{\mathbf{R}}(\hat{\mathbf{R}}), \boldsymbol{\xi}_{ik} \right\rangle \cdot \left\langle \text{Log}_{\mathbf{R}}(\hat{\mathbf{R}}), \boldsymbol{\xi}_{j\ell} \right\rangle \right\}, \quad (6.2)$$

where we used the same indexing convention as for the FIM. Of course, all  $d \times d$  blocks  $(i, j)$  such that either  $i$  or  $j$  or both are in  $A$  are zero by construction. In particular, the variance of  $\hat{\mathbf{R}}$  is the trace of  $C_A$ :

$$\text{var}_A \triangleq \text{trace } C_A = \mathbb{E} \left\{ \|\text{Log}_{\mathbf{R}}(\hat{\mathbf{R}})\|^2 \right\} = \mathbb{E} \left\{ \text{dist}^2(\mathbf{R}, \hat{\mathbf{R}}) \right\}, \quad (6.3)$$

where  $\text{dist}$  is the geodesic distance on  $\mathcal{P}_A$  (3.16).

In order to use [7, Theorem 4], we need to compute  $F_A \triangleq P_A F P_A$ , where  $P_A$  is the orthogonal projector from  $\text{T}_{\mathbf{R}}\mathcal{P}$  to  $\text{T}_{\mathbf{R}}\mathcal{P}_A$ , expressed w.r.t. the orthonormal basis (5.6). The effect of  $P_A$  is to set all rows and columns corresponding to anchored rotations to zero. Thus, we introduce the masked Laplacian  $\mathcal{L}_A$ :

$$(\mathcal{L}_A)_{ij} = \begin{cases} \mathcal{L}_{ij} & \text{if } i, j \notin A, \\ 0 & \text{otherwise.} \end{cases} \quad (6.4)$$

Then, the projected FIM is simply:

$$F_A = \frac{1}{d} (\mathcal{L}_A \otimes I_d). \quad (6.5)$$

The pseudoinverse of  $F_A$  is given by  $F_A^\dagger = d(\mathcal{L}_A^\dagger \otimes I_d)$ , since for arbitrary matrices  $A$  and  $B$ , it holds that  $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$  [7, Prop. 6]. Notice that the rows and columns of  $\mathcal{L}_A^\dagger$  corresponding to anchors are also zero. The aforementioned theorem [7, Thm. 4] then yields the sought CRB:

$$C_A \succeq d(\mathcal{L}_A^\dagger \otimes I_d) + \text{curvature terms}. \quad (6.6)$$

We compute the curvature terms explicitly in Appendix C and show that they are on the order of  $\text{SNR}^{-2}$ . They can hence be neglected for large SNR, which makes for easier formulas. In particular, for  $n = 2$ , the manifold  $\mathcal{P}_A$  is flat. Hence, the curvature terms vanish exactly and the CRB takes up the form:

$$C_A \succeq \mathcal{L}_A^\dagger. \quad (6.7)$$

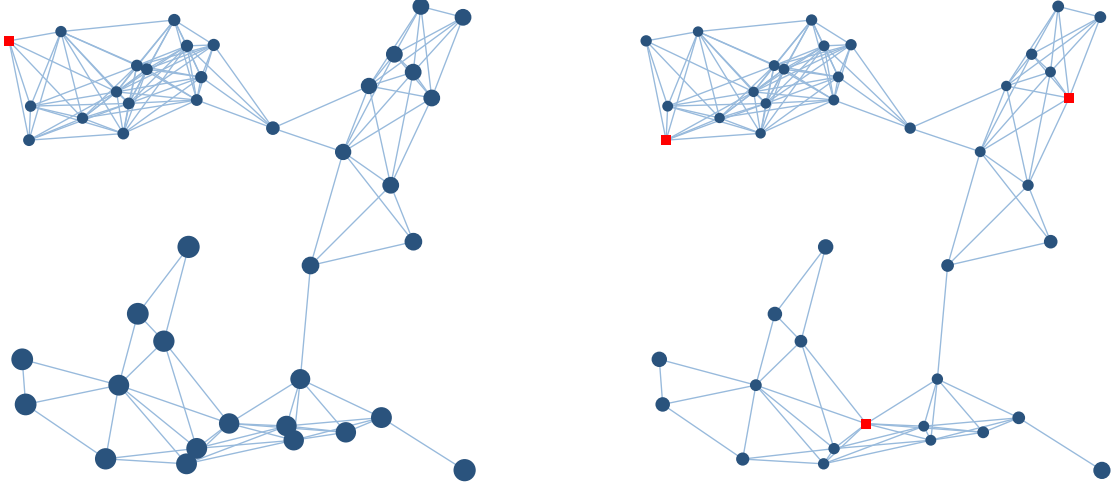


Figure 1: The Cramér-Rao bound for anchored synchronization (6.10) limits how well each individual rotation can be estimated. We illustrate the effect of anchors on the two identical synchronization graphs above, where all edges have the same weight (i.i.d. noise). Anchors are red squares. Unknown rotations are the blue round nodes. The area of node  $i$  is proportional to the lower-bound on the average error for this node  $\mathbb{E} \left\{ \text{dist}^2(R_i, \hat{R}_i) \right\}$ . On the left, there is only one anchor in the upper-left cluster. Hence, nodes in the lower-left cluster, which are separated from the anchor by two bottlenecks, will be seriously harder to estimate accurately than in the situation on the right, where there is one anchor for each cluster. This (obvious) fact is supported by precise, quantified bounds, making it possible to automate network analysis and design for large graphs.

For  $n = 3$ , including the curvature terms yields this CRB:

$$C_A \succeq 3 \left( \mathcal{L}_A^\dagger - \frac{1}{4} \left( \text{ddiag}(\mathcal{L}_A^\dagger) \mathcal{L}_A^\dagger + \mathcal{L}_A^\dagger \text{ddiag}(\mathcal{L}_A^\dagger) \right) \right) \otimes I_3, \quad (6.8)$$

where  $\text{ddiag}$  sets all off-diagonal entries of a matrix to zero. The curvature terms are thus on the order of  $\mathcal{O}(\lambda_{\max}^2(\mathcal{L}_A^\dagger))$ , which indeed becomes small at large SNR. For general  $n$ , neglecting curvature for  $n \geq 3$ , the variance is lower-bounded as follows:

$$\text{var}_A = \mathbb{E} \left\{ \text{dist}^2(\mathbf{R}, \hat{\mathbf{R}}) \right\} \geq d^2 \text{trace } \mathcal{L}_A^\dagger, \quad (6.9)$$

where  $\text{dist}$  is as defined by (3.16). It also holds for each node  $i$  that

$$\mathbb{E} \left\{ \text{dist}^2(R_i, \hat{R}_i) \right\} \geq d^2 (\mathcal{L}_A^\dagger)_{ii}. \quad (6.10)$$

This leads to a useful interpretation of the CRB depicted in Figure 1.

Interestingly, for synchronization on the group of translations  $\mathbb{R}^n$ , the CRB involves the same Laplacian-related matrix [7]. Barooah and Hespanha give interpretations of this bound in terms of the resistance distance on the measurement graph when there is exactly one anchor [5]. This analogy is notably useful in verifying important properties of the bound, such as its behavior when edges or nodes are added or deleted, etc.

## 6.2 Anchor-free synchronization

When no anchors are provided, the rotation matrices  $R_i$  can only be recovered up to a global rotation, leading to an equivalence relation (3.17) on  $\mathcal{P}$ , which in turn leads us to work on the Riemannian quotient parameter space  $\mathcal{P}_\emptyset$  (3.18). The synchronization problem is well-posed on  $\mathcal{P}_\emptyset$  as long as the measurement graph is connected, which we always assume in this work.

We set out to use [7, Theorem 5], which states Cramér-Rao bounds for estimation problems on Riemannian quotient manifolds. To this end, we first define the  $dN \times dN$  covariance matrix  $C_\emptyset$  for anchor-free synchronization, then we state the bound.

**Definition 4** (anchor-free covariance). *Following [7, eq. (25)], the covariance matrix of an estimator  $[\hat{\mathbf{R}}]$  mapping each possible set of measurements  $H_{ij}$  to a point in  $\mathcal{P}_\emptyset$  (that is, to an equivalence class in  $\mathcal{P}$ ), expressed w.r.t. the orthonormal basis (5.6) of  $\mathbf{T}_{\mathbf{R}}\mathcal{P}$ , is given by:*

$$(C_\emptyset)_{ij,k\ell} = \mathbb{E} \{ \langle \boldsymbol{\xi}, \boldsymbol{\xi}_{ik} \rangle \cdot \langle \boldsymbol{\xi}, \boldsymbol{\xi}_{j\ell} \rangle \}, \text{ with} \quad (6.11)$$

$$\boldsymbol{\xi} = (\mathbf{D}\pi(\mathbf{R})|_{\mathbf{H}_{\mathbf{R}}})^{-1} [\text{Log}_{[\mathbf{R}]}([\hat{\mathbf{R}}])]. \quad (6.12)$$

That is,  $\boldsymbol{\xi}$  (the random error vector) is the shortest horizontal vector such that  $\text{Exp}_{\mathbf{R}}(\boldsymbol{\xi}) \in [\hat{\mathbf{R}}]$ . We used the same indexing convention as for the FIM. In particular, the variance of  $[\hat{\mathbf{R}}]$  is the trace of  $C_\emptyset$ :

$$\text{var}_\emptyset \triangleq \text{trace } C_\emptyset = \mathbb{E} \left\{ \|\text{Log}_{[\mathbf{R}]}([\hat{\mathbf{R}}])\|^2 \right\} = \mathbb{E} \left\{ \text{dist}^2([\mathbf{R}], [\hat{\mathbf{R}}]) \right\}, \quad (6.13)$$

where  $\text{dist}$  is the geodesic distance on  $\mathcal{P}_\emptyset$  (3.30).

Theorem [7, Thm. 5] then yields the sought CRB:

$$C_\emptyset \succeq d(\mathcal{L}^\dagger \otimes I_d) + \text{curvature terms}. \quad (6.14)$$

We compute the curvature terms explicitly in Appendix C and show they can be neglected for (reasonably) large enough SNR. In particular, for  $n = 2$ , the manifold  $\mathcal{P}_\emptyset$  is flat. Hence:

$$C_\emptyset \succeq \mathcal{L}^\dagger. \quad (6.15)$$

For  $n = 3$ , the curvature terms are the same as those for the anchored case, with an additional term that decreases as  $1/N$ . For (not so) large  $N$  then, the bound (6.8) is a good bound for  $n = 3$ , anchor-free synchronization. For general  $n$ , neglecting curvature for  $n \geq 3$ , the variance is lower-bounded as follows:

$$\text{var}_\emptyset = \mathbb{E} \left\{ \text{dist}^2([\mathbf{R}], [\hat{\mathbf{R}}]) \right\} \geq d^2 \text{trace } \mathcal{L}^\dagger, \quad (6.16)$$

where  $\text{dist}$  is as defined by (3.30).

For the remainder of this section, we work out an interpretation of (6.14). This matrix inequality entails that, for all  $x \in \mathbb{R}^{dN}$  (neglecting curvature terms if needed):

$$x^\top C_\emptyset x \geq d x^\top (\mathcal{L}^\dagger \otimes I_d) x. \quad (6.17)$$

As both the covariance and the FIM correspond to positive semidefinite operators on the horizontal space  $\mathbf{H}_{\mathbf{R}}$ , this is really only meaningful when  $x$  is the vector of coordinates of a horizontal vector  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_N) \in \mathbf{H}_{\mathbf{R}}$ . Let  $e_i, e_j$  denote the  $i$ -th and  $j$ -th columns of the identity matrix  $I_N$  and let  $e_k$  denote the  $k$ -th column of  $I_d$ . We consider  $x = (e_i - e_j) \otimes e_k$ , which corresponds to the zero horizontal vector  $\boldsymbol{\eta}$  except for  $\eta_i = R_i E_k$  and  $\eta_j = -R_j E_k$ , with  $E_k \in \mathfrak{so}(n)$  the  $k$ -th element of the orthonormal basis of  $\mathfrak{so}(n)$  picked as in (5.5). By definition of  $C_\emptyset$  and of the error vector  $\boldsymbol{\xi} = (R_1 \Omega_1, \dots, R_N \Omega_N) \in \mathbf{H}_{\mathbf{R}}$  (6.12),

$$x^\top C_\emptyset x = \mathbb{E} \left\{ \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle^2 \right\} = \mathbb{E} \left\{ \langle \Omega_i - \Omega_j, E_k \rangle^2 \right\}. \quad (6.18)$$

On the other hand, we have

$$d x^\top (\mathcal{L}^\dagger \otimes I_d) x = d (e_i - e_j)^\top \mathcal{L}^\dagger (e_i - e_j). \quad (6.19)$$

These two last quantities are related by inequality (6.17). Summing for  $k = 1 \dots d$  on both sides of this inequality, we find:

$$\mathbb{E} \left\{ \|\Omega_i - \Omega_j\|_{\mathbf{F}}^2 \right\} \geq d^2 (e_i - e_j)^\top \mathcal{L}^\dagger (e_i - e_j). \quad (6.20)$$



Now remember that the error vector  $\boldsymbol{\xi}$  (6.12) is the shortest horizontal vector such that  $\text{Exp}_{\mathbf{R}}(\boldsymbol{\xi}) \in [\hat{\mathbf{R}}]$ . Without loss of generality, we may assume that  $\hat{\mathbf{R}}$  is aligned such that  $\text{Exp}_{\mathbf{R}}(\boldsymbol{\xi}) = \hat{\mathbf{R}}$ . Then,  $\hat{R}_i = R_i \exp(\Omega_i)$  for all  $i$ . It follows that

$$\hat{R}_i \hat{R}_j^\top = R_i \exp(\Omega_i) \exp(-\Omega_j) R_j^\top, \quad \text{hence} \quad (6.21)$$

$$\text{dist}^2(R_i R_j^\top, \hat{R}_i \hat{R}_j^\top) = \|\log(\exp(\Omega_i) \exp(-\Omega_j))\|_{\text{F}}^2. \quad (6.22)$$

For commuting  $\Omega_i$  and  $\Omega_j$ —which is always the case for  $n = 2$ —we have

$$\log(\exp(\Omega_i) \exp(-\Omega_j)) = \Omega_i - \Omega_j. \quad (6.23)$$

For  $n \geq 3$ , this still approximately holds at high enough SNR (that is, for small enough  $\Omega_i, \Omega_j$ ), by the Baker-Campbell-Hausdorff formula. Hence,

$$\mathbb{E} \left\{ \text{dist}^2(R_i R_j^\top, \hat{R}_i \hat{R}_j^\top) \right\} \approx \mathbb{E} \left\{ \|\Omega_i - \Omega_j\|_{\text{F}}^2 \right\} \geq d^2 (e_i - e_j)^\top \mathcal{L}^\dagger (e_i - e_j). \quad (6.24)$$

The quantity  $(e_i - e_j)^\top \mathcal{L}^\dagger (e_i - e_j)$  is sometimes called the squared Euclidean Commute Time Distance (ECTD) [32] between nodes  $i$  and  $j$  (up to a normalization constant). It is also known as the electrical resistance distance. It is rich in interpretation: the ECTD between two nodes is inversely proportional to the ease of communication between these two nodes. It decreases whenever the number of paths between them increases or whenever an existing path is made more informative, i.e., weights on that path are increased. Put this in perspective considering that in our setting, weights are proportional to the quality of the measurements.

Still in [32], it is shown in Section 5 how the ECTD can be used to embed the nodes in the plane such that two nodes are close-by if they have small ECTD. For synchronization, such an embedding naturally groups together nodes whose relative rotations can be accurately estimated, as depicted in Figure 2.

The CRB for anchor-free synchronization thus sets a lower-bound on how well one can estimate the relative rotation between  $R_i$  and  $R_j$ . The bound reflects how well the corresponding nodes are connected in the information-weighted measurement graph. We emphasize that the anchor-free CRB does not say anything about any particular rotation.

## 7 Comments on, and consequences of the CRB

Neglecting curvature terms—which we showed is exact for  $n = 2$  and legitimate for  $n \geq 3$  at reasonably favorable SNR—the Cramér-Rao bounds yield simple lower-bounds on the variance of unbiased estimators for synchronization. The right-hand sides of these bounds could in turn be used as cost function in an effort to optimize network structure or anchor placement. We take a closer look at the anchor-free bound to determine what structural properties it is affected by.

At large SNR, the anchor-free bound, normalized by the number of rotations  $N$ , reads (6.16):

$$\mathbb{E} \{ \text{MSE} \} \triangleq \mathbb{E} \left\{ \frac{1}{N} \text{dist}^2([\mathbf{R}], [\hat{\mathbf{R}}]) \right\} \geq \frac{d^2}{N} \text{trace}(\mathcal{L}^\dagger), \quad (7.1)$$

where  $\mathbb{E} \{ \text{MSE} \}$  as defined is the *expected mean squared error* of an unbiased estimator  $[\hat{\mathbf{R}}]$ . This expression shows the limiting role of the trace of the pseudoinverse of the information-weighted Laplacian  $\mathcal{L}$  (5.21) of the measurement graph. This role has been established before for other synchronization problems for simpler groups and simpler noise models [7, 21]. We now shed some light on this result by stating a few elementary consequences of it. We let

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N \quad (7.2)$$

denote the eigenvalues of  $\mathcal{L}$ , where  $\lambda_2 > 0$  means the measurement graph is assumed connected.

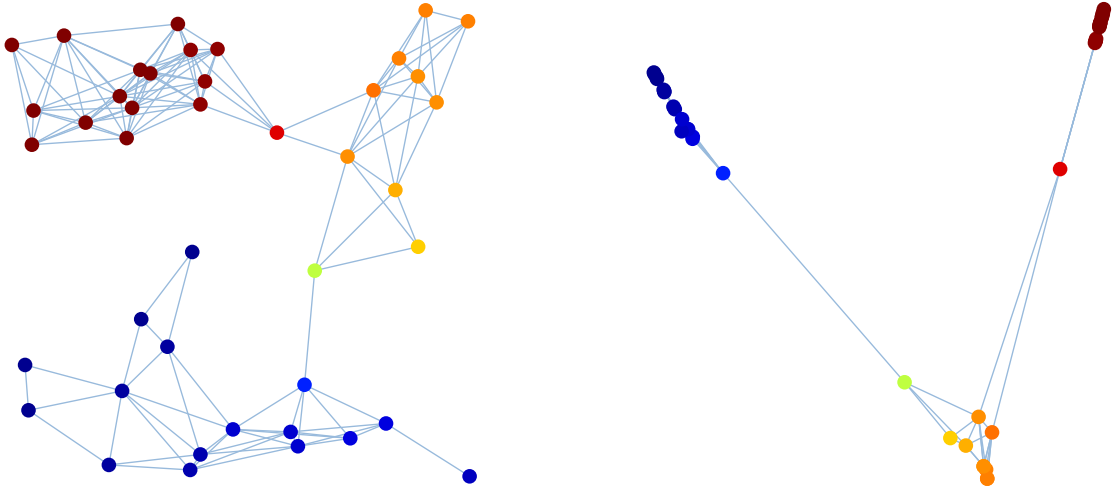


Figure 2: The Fisher information matrix for anchor-free synchronization is essentially the Laplacian of the measurement graph, whose edges have weights proportional to the amount of information these edges contain (5.22). As a result, the Cramér-Rao bound limits the accuracy to which the relative rotation between two nodes can be estimated in proportion to the Euclidean commute time distance (ECTD) separating them in the graph (6.24) [32]. On the left, a synchronization graph: each node corresponds to a rotation to estimate, all edges correspond to measurements of relative rotations affected by i.i.d. noise, and hence have the same weight. Nodes are colored according to the Fiedler vector of the weighted graph. Node positions are irrelevant. On the right, ECTD-embedding of the same graph in the plane, such that the distance between two nodes  $i$  and  $j$  in the picture is mostly proportional to the ECTD separating them, which is essentially a lower-bound on  $\mathbb{E} \left\{ \text{dist}^2(R_i R_j^\top, \hat{R}_i \hat{R}_j^\top) \right\}$ . In other words: the closer two nodes are, the better their relative rotation can be estimated.

**A larger Fiedler value is better.** The right-hand side (rhs) of (7.1) in terms of the  $\lambda_i$ 's is given by (omitting the unimportant  $d^2$  factor):

$$\frac{1}{N} \text{trace}(\mathcal{L}^\dagger) = \frac{1}{N} \sum_{i=2}^N \frac{1}{\lambda_i} \leq \frac{N-1}{N} \frac{1}{\lambda_2} \approx \frac{1}{\lambda_2} \text{ (for large } N\text{)}. \quad (7.3)$$

The second eigenvalue  $\lambda_2$  is known as the *Fiedler value* (or *algebraic connectivity*) of the information-weighted measurement graph. It is well known that the Fiedler value is low in the presence of bottlenecks in the graph and high in the presence of many, heavy spanning trees. The latter equation translates in the following intuitive statement: by increasing the Fiedler value of the measurement graph, one can force the rhs of the CRB (7.1) down. Not surprisingly then, expander graphs are ideal for synchronization, since, by design, their Fiedler value  $\lambda_2$  is bounded away from zero while simultaneously being sparse [20].

Notice that the Fiedler vector has zero mean (it is orthogonal to  $\mathbf{1}_N$ ) and hence describes the horizontal vectors of maximum variance. It is thus also the first axis of the graph on the right in Figure 2.

**The CRB is an asymptotic bound.** Intrinsic Cramér-Rao bounds are asymptotic bounds, that is, they are meaningful for small errors. In particular, in the present scenario the parameter spaces  $\mathcal{P}_A$  and  $\mathcal{P}_\theta$  are compact, hence there is an upper-bound on the variance of any estimator. The CRB is unable to capture such a global feature, and hence for arbitrarily low SNR, the CRB will predict an arbitrarily large variance (and will be violated). As a means to locate the point at which the CRB certainly stops making sense, we consider the problem of estimating a rotation matrix  $R \in \text{SO}(n)$  based on a measurement  $Z \in \text{SO}(n)$  of  $R$ , and compute the variance of the (unbiased) estimator  $\hat{R}(Z) := Z$  when  $Z$  is uniformly random, i.e., when no information is available.

Define  $V_n = \mathbb{E} \{ \text{dist}^2(Z, R) \}$  for  $Z \sim \text{Uni}(\text{SO}(n))$ . A computation using Weyl's formula yields:

$$V_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|\log(Z^\top R)\|_F^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\theta^2 d\theta = \frac{2\pi^2}{3}, \quad V_3 = \frac{2\pi^2}{3} + 4. \quad (7.4)$$

A reasonable upper-bound on the variance of an estimator should be  $N'V_n$ , where  $N'$  is the number of independent rotations to estimate ( $N - 1$  for anchor-free synchronization,  $N - |A|$  for anchored synchronization). A CRB larger than this should be disregarded.

**trace( $\mathcal{L}^\dagger$ ) plays a limiting role in synchronization.** The quantity  $\text{trace}(\mathcal{L}^\dagger)$  appears naturally in CRB's for synchronization problems on groups [7, 21]. For complete graphs and constant weight  $w$ ,  $\text{trace}(\mathcal{L}^\dagger) = \frac{N-1}{wN} \approx \frac{1}{w}$ . Then, by (7.1),

$$\mathbb{E} \{ \text{MSE} \} \gtrsim \frac{d^2}{wN}. \quad (7.5)$$

If the measurement graph is sampled from a distribution of random graphs,  $\text{trace}(\mathcal{L}^\dagger)$  becomes a random variable. We feel that the study of this random variable for various families of random graph models, such as Erdős-Rényi, small-world or scale-free graphs [22] is a question of interest, probably best addressed using the language of random matrix theory.

Let us consider Erdős-Rényi graphs  $G_{N,q}$  with  $N$  nodes and edge density  $q \in (0, 1)$ , that is, graphs such that any edge is present with probability  $q$ , independently from the other edges. Let all the edges have equal weight  $w$ . Let  $\mathcal{L}_{N,q}$  be the Laplacian of a  $G_{N,q}$  graph. The expected Laplacian is  $\mathbb{E} \{ \mathcal{L}_{N,q} \} = wq(NI_N - \mathbf{1}_{N \times N})$ , which has eigenvalues  $\lambda_1 = 0, \lambda_2 = \dots = \lambda_N = Nwq$ . Hence,  $\text{trace}(\mathbb{E} \{ \mathcal{L}_{N,q} \}^\dagger) = \frac{N-1}{N} \frac{1}{wq}$ . A more useful statement can be made using [8, Thm. 1.4] and [16, Thm. 2]. These theorems state that, asymptotically as  $N$  grows to infinity, all eigenvalues of  $\mathcal{L}_{N,q}/N$  converge to  $wq$  (except of course for one zero eigenvalue). Consequently (details omitted),

$$\lim_{N \rightarrow \infty} \text{trace}(\mathcal{L}_{N,q}^\dagger) = \frac{1}{wq} \text{ (in probability)}. \quad (7.6)$$

For large  $N$ , we use the approximation  $\text{trace}(\mathcal{L}_{N,q}^\dagger) \approx 1/wq$ . Then, by (7.1), for large  $N$  we have:

$$\mathbb{E} \{ \text{MSE} \} \gtrsim \frac{d^2}{wqN}. \quad (7.7)$$

Notice how for fixed measurement quality  $w$  and density  $q$ , the lower-bound on the expected MSE decreases with the number  $N$  of rotations to estimate.

**Synchronization can withstand many outliers.** Consider the Langevin with outliers distribution from Example 5, where (on average) a fraction  $1 - p$  of measurements are sampled uniformly at random, i.e., they are outliers. The information weight  $w(p) = \alpha_n(\kappa, p)$  for some fixed concentration  $\kappa > 0$  is given by equations (5.28) and (5.29) for  $n = 2$  and 3 respectively. A Taylor development around  $p = 0$  shows that

$$w(p) = a_{n,\kappa}p^2 + \mathcal{O}(p^3) \quad (7.8)$$

for some positive constant  $a_{n,\kappa}$ . Then, for  $p \ll 1$ , building upon (7.5) for complete graphs with i.i.d. measurements we get:

$$\mathbb{E} \{ \text{MSE} \} \gtrsim \frac{d^2}{a_{n,\kappa}p^2N}. \quad (7.9)$$

If one needs to get the rhs of this inequality down to a tolerance  $\varepsilon^2$ , the probability  $p$  of a measurement not being an outlier needs to be at least as large as:

$$p_\varepsilon \triangleq \frac{d}{\sqrt{a_{n,\kappa}\varepsilon}} \frac{1}{\sqrt{N}}. \quad (7.10)$$

If  $p_\varepsilon > 1$ , synchronization cannot be solved to the desired accuracy. The  $1/\sqrt{N}$  factor is the most interesting: it establishes that as the number of nodes increases, synchronization can withstand more and more outliers.

This result is to be put in perspective with the bound in [34, eq. (37)] for  $n = 2$ ,  $\kappa = \infty$ , where it is shown that as soon as  $p > 1/\sqrt{N}$ , there is enough information in the measurements (on average) for their eigenvector method to do better than random synchronization. It is also shown there that, as  $p^2 N$  goes to infinity, the correlation between the eigenvector estimator and the true rotations goes to 1. Similarly, we see that as  $p^2 N$  increases to infinity, the rhs of the CRB goes to zero. Our analysis further shows that the role of  $p^2 N$  is tied to the problem itself (not to a specific estimation algorithm), and remains the same for  $n > 2$  and in the presence of Langevin noise on the good measurements.

Building upon (7.6) where it is stated that  $\text{trace}(\mathcal{L}^\dagger) \approx 1/wq = N(N-1)/(2Mw)$  for Erdős-Rényi graphs with  $N$  nodes and  $M$  edges, we define  $p_\varepsilon$  for such graphs as:

$$p_\varepsilon \triangleq \frac{d}{\sqrt{a_{n,\kappa}}\varepsilon} \sqrt{\frac{N}{2M}}. \quad (7.11)$$

To conclude this remark, we provide numerically computable expressions for  $a_{n,\kappa}$ ,  $n = 2$  and 3 and give an example:

$$a_{2,\kappa} = \frac{\kappa^2}{\pi c_2^2(\kappa)} \int_0^\pi (1 - \cos 2\theta) \exp(4\kappa \cos \theta) d\theta, \quad (7.12)$$

$$a_{3,\kappa} = \frac{\kappa^2 e^{2\kappa}}{\pi c_3^2(\kappa)} \int_0^\pi (1 - \cos 2\theta)(1 - \cos \theta) \exp(4\kappa \cos \theta) d\theta. \quad (7.13)$$

As an example, we generated an ER graph with  $N = 2500$  nodes and edge density of 60% for synchronization of rotations in  $\text{SO}(3)$  with i.i.d. noise following a  $\text{LangUni}(I, \kappa = 7, p)$ . The CRB (7.1), which requires complete knowledge of the graph to compute  $\text{trace}(\mathcal{L}^\dagger)$ , tells us that we need  $p \geq 2.1\%$  to reach an accuracy level of  $\varepsilon = 10^{-1}$  (for comparison,  $\varepsilon^2$  is roughly 1000 times smaller than  $V_3$  (7.4)). The simple formula (7.11), which can be computed quickly solely based on the graph statistics  $N$  and  $M$ , yields  $p_\varepsilon = 2.2\%$ .

## 8 Conclusions and perspectives

In this work, we considered the problem of estimating a set of rotations  $R_i \in \text{SO}(n)$  based on pairwise measurements of relative rotations  $R_i R_j^\top$ . We provided a framework to study synchronization as estimation on manifolds for arbitrary  $n$  under a large family of noise models. We established formulas for the Fisher information matrix and associated Cramér-Rao bounds of synchronization and provided interpretation and visualization tools for them in both the anchored and anchor-free scenarios. In the analysis of these bounds, we notably pointed out the high robustness of synchronization against random outliers.

Because of the crucial role of the pseudoinverse of the Laplacian  $\mathcal{L}^\dagger$  of weighted graphs (and their traces) in the CRB's we established, it would be interesting to study efficient methods to compute such objects, see e.g. [23]. Likewise, exploring the distribution of  $\text{trace}(\mathcal{L}^\dagger)$  seen as a random variable for various models of random graphs should bring some insight as to which networks are naturally easy to synchronize. Expander graphs already emerge as good candidates.

The Laplacian of the measurement graph plays the same role in bounds for synchronization of rotations as for synchronization of translations. Carefully checking the proof given in the present work, it is reasonable to speculate that the Laplacian would appear similarly in CRB's for synchronization on any Lie group, as long as we assume independence of noise affecting different measurements and some symmetry in the noise distribution. Such a generalization would in particular yield CRB's for synchronization on the special Euclidean group of rigid body motions,  $\mathbb{R}^3 \rtimes \text{SO}(3)$ .

In future work, we will leverage the formulation of synchronization as an estimation problem on manifolds to propose maximum-likelihood estimators for synchronization. Such approaches will

result in optimization problems on the parameter manifolds whose geometries we described here. By executing the derivations with the Langevin + outliers noise model, this effort should lead to naturally robust synchronization algorithms.

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## A Registration on $\text{SO}(n)$

Let  $Q, Q' \in \text{SO}(n)$ . The squared Riemannian distance on  $\text{SO}(n)$  between  $Q$  and  $Q'$  is:

$$\text{dist}^2(Q, Q') = \|\log(Q'Q^\top)\|_F^2 = \|\log(Q^\top Q')\|_F^2. \quad (\text{A.1})$$

By extension, let  $\mathbf{R} = (R_1, \dots, R_N) \in \mathcal{P} = \text{SO}(n) \times \dots \times \text{SO}(n)$  and  $\mathbf{R}' = (R'_1, \dots, R'_N) \in \mathcal{P}$ . The squared Riemannian distance between  $\mathbf{R}$  and  $\mathbf{R}'$  on  $\mathcal{P}$  is

$$\text{dist}^2(\mathbf{R}, \mathbf{R}') = \sum_{i=1}^N \text{dist}^2(R_i, R'_i). \quad (\text{A.2})$$

We use the notation  $\mathbf{R}Q \triangleq (R_1Q, \dots, R_NQ) \in \mathcal{P}$ . Recall that  $\mathcal{P}_\emptyset$  is the quotient manifold defined as  $\mathcal{P}_\emptyset = \mathcal{P} / \sim$ , with the equivalence relation  $\mathbf{R} \sim \mathbf{R}' \Leftrightarrow \exists Q \in \text{SO}(n) : \mathbf{R} = \mathbf{R}'Q$ . Then, the squared Riemannian distance between the equivalence classes  $[\mathbf{R}]$  and  $[\mathbf{R}']$  is defined in terms of the distance on  $\mathcal{P}$  as:

$$\text{dist}^2([\mathbf{R}], [\mathbf{R}']) = \min_{Q, Q' \in \text{SO}(n)} \text{dist}^2(\mathbf{R}Q, \mathbf{R}'Q'). \quad (\text{A.3})$$

That is, the distance between the two sets  $[\mathbf{R}]$  and  $[\mathbf{R}']$  is the distance separating two of the closest points  $\mathbf{R}Q \in [\mathbf{R}]$  and  $\mathbf{R}'Q' \in [\mathbf{R}']$  of these sets. This is sometimes referred to as *registration* on  $\text{SO}(n)$ , as our goal is to align  $\mathbf{R}$  and  $\mathbf{R}'$  such that they look as similar as possible. One only needs to optimize over either  $Q$  or  $Q'$ . Choosing to optimize over  $Q$ , we further develop the latter expression:

$$\text{dist}^2([\mathbf{R}], [\mathbf{R}']) = \min_{Q \in \text{SO}(n)} \text{dist}^2(Q, \mathbf{R}') = \min_{Q \in \text{SO}(n)} \sum_{i=1}^N \text{dist}^2(R_iQ, R'_i) \quad (\text{A.4})$$

$$= \min_{Q \in \text{SO}(n)} \sum_{i=1}^N \|\log(Q^\top R_i^\top R'_i)\|_F^2 = \min_{Q \in \text{SO}(n)} \sum_{i=1}^N \text{dist}^2(Q, R_i^\top R'_i). \quad (\text{A.5})$$

Hence, an optimal  $Q$  is a *Karcher mean*—or *intrinsic mean* or *Riemannian center of mass*—of the rotation matrices  $R_1^\top R'_1, \dots, R_N^\top R'_N$ .

Although the minimum value is always well-defined because  $\text{SO}(n)$  is compact, the optimal rotation  $Q$  may, in general, not be unique. Manton [24] gives a simple condition under which the Karcher mean on compact Lie groups is unique, and gives a convergent algorithm to compute it. On  $\text{SO}(n)$ , this result reads:

**Theorem A.1.** (Manton 2004) Let  $R_1^\top R'_1, \dots, R_N^\top R'_N \in \text{SO}(n)$ . If there exists  $Y \in \text{SO}(n)$  and  $0 < r < \pi/2$  such that  $\text{dist}(Y, R_i^\top R'_i) < r$  for  $i = 1 \dots N$ , then (i) the Karcher mean, i.e., the global minimizer of (A.5), is unique, and (ii) the sequence  $(Q_k)_{k=0,1,2,\dots}$  defined by

$$Q_{k+1} = Q_k \exp \left( \frac{1}{N} \sum_{i=1}^N \log(Q_k^\top R_i^\top R'_i) \right) \quad (\text{A.6})$$

with initial condition  $Q_0 = R_1^\top R'_1$  converges to the global minimizer.

*Proof.* See [24, Thm. 5]. □

An easy—but possibly inconclusive—way of checking the condition a posteriori after  $K$  iterations is to take  $Y := Q_K$  in the theorem condition and verifying that  $\text{dist}(Y, R_i^\top R'_i) < \pi/2$  for  $i = 1 \dots N$ . This result may not be optimal, and faster algorithms certainly exist. For more information regarding Karcher means, see [2, 27, 30].

For  $n = 2$ , Since  $\text{SO}(2)$  is Abelian (the product of rotations in the plane is commutative), the Karcher mean is easy to compute. Indeed, for commuting matrices  $A, B$ , we have  $\log(AB) = \log(A) + \log(B)$  and  $\exp(A+B) = \exp(A)\exp(B)$ . Then, the algorithm in Theorem A.1 converges in one iteration. Forcing  $Q_0 = I$  yields an analytical formula for the Karcher mean of rotations in the plane. For  $\text{SO}(3)$ , the maximum distance between two rotations is  $\sqrt{2}\pi$ , hence the uniqueness condition in Theorem A.1 is not always satisfied, but is rather permissive in the context of our application.

A simpler approach to registration uses the Frobenius norm in the embedding space  $\mathbb{R}^{n \times n}$  of  $\text{SO}(n)$  instead of Riemannian distances:

$$\text{embdist}^2([\mathbf{R}], [\mathbf{R}']) \triangleq \min_{Q \in \text{SO}(n)} \sum_{i=1}^N \|R_i Q - R'_i\|_F^2 = \min_{Q \in \text{SO}(n)} \sum_{i=1}^N \|Q - R_i^\top R'_i\|_F^2. \quad (\text{A.7})$$

The second equality holds since the Frobenius norm is invariant under orthogonal transformations. The optimal  $Q$ —which we now refer to simply as  $Q$ —is what is sometimes called the *projected Euclidean mean* of the  $N$  rotation matrices  $R_i^\top R'_i$ ,  $i = 1 \dots N$ . Indeed, ignoring the constraint  $Q \in \text{SO}(n)$ , the solution would be given by  $P \in \mathbb{R}^{n \times n}$ :

$$P = \frac{1}{N} \sum_{i=1}^N R_i^\top R'_i. \quad (\text{A.8})$$

A number of authors [27, 33] provide closed-form expressions for  $Q$  and study its existence and uniqueness. Reformulating [33, Prop. 3.3] in terms of the SVD decomposition of  $P$ , we may express  $Q$  in all cases as:

$$P = USV^\top \text{ (SVD decomposition)}, \quad J = \begin{pmatrix} I_{n-1} & \\ & \det(UV^\top) \end{pmatrix}, \quad Q = UJV^\top, \quad (\text{A.9})$$

which essentially comes down to normalizing the singular values of  $P$  so as to make it orthogonal, and ensuring the result has determinant  $+1$ . The rotation matrix  $Q$  is uniquely defined by (A.9) if either of these conditions is met:

(i)  $\det(UV^\top) = +1$  and the multiplicity of 0 as a singular value of  $P$  is at most 1.

(ii)  $\det(UV^\top) = -1$  and the multiplicity of the smallest singular value of  $P$  is 1.

Substituting (A.9) in (A.7), we get:

$$\text{embdist}^2([\mathbf{R}], [\mathbf{R}']) = \sum_{i=1}^N \|R_i Q - R'_i\|_F^2 = 2Nn - 2\text{trace} \left( Q^\top \sum_{i=1}^N R_i^\top R'_i \right) \quad (\text{A.10})$$

$$= 2Nn - 2N\text{trace}(JS) \quad (\text{A.11})$$

$$= 2N(n - \sigma_1 - \sigma_2 - \dots - \sigma_{n-1} - s\sigma_n), \quad (\text{A.12})$$



where  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$  are the singular values of  $P$  and  $s = \det(UV^\top)$ .

Registration with the embedded distance has the advantage of being straightforward to compute and uniquely defined in all but pathological cases. Likewise, there is a natural embedded distance for  $\mathcal{P}_A$ :

$$\text{embdist}^2(\mathbf{R}, \mathbf{R}') \triangleq \sum_{i \notin A} \|R_i - R'_i\|_{\mathbb{F}}^2. \quad (\text{A.13})$$

## B Density normalization

This appendix presents the derivation of the normalization coefficients  $c_n(\kappa)$  (4.6)–(4.8) that appear in the probability density function (4.4). Recall that the coefficient  $c_n(\kappa)$  is given by (4.5):

$$c_n(\kappa) = \int_{\text{SO}(n)} \exp(\kappa \text{trace}(Z)) \, d\mu(Z), \quad (\text{B.1})$$

where  $d\mu$  is the normalized Haar measure over  $\text{SO}(n)$  such that  $\int_{\text{SO}(n)} d\mu(Z) = 1$ . In particular,  $c_n(0) = 1$ . The integrand,  $g(Z) = \exp(\kappa \text{trace}(Z))$ , is a class function, meaning that for all  $Q, Z \in \text{SO}(n)$  we have  $g(Z) = g(QZQ^{-1})$ . We are thus in a position to use the Weyl integration formula specialized to  $\text{SO}(n)$  [9, Exercise 18.1–2]. Formula (4.3) applies,

$$\int_{\text{SO}(4)} g(Z) \, d\mu(Z) = \frac{1}{4(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \tilde{g}(\theta_1, \theta_2) \cdot |e^{i\theta_1} - e^{i\theta_2}|^2 \cdot |e^{i\theta_1} - e^{-i\theta_2}|^2 \, d\theta_1 d\theta_2, \quad (\text{B.2})$$

where we defined

$$\tilde{g}(\theta_1, \theta_2) \triangleq g \left( \text{diag} \left( \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}, \begin{pmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{pmatrix} \right) \right). \quad (\text{B.3})$$

This reduces the problem to a classical integral over the square—or really the torus— $[-\pi, \pi] \times [-\pi, \pi]$ . Evaluating  $\tilde{g}$  is straightforward:

$$\tilde{g}(\theta_1, \theta_2) = \exp(2\kappa \cdot [\cos \theta_1 + \cos \theta_2]). \quad (\text{B.4})$$

Using trigonometric identities, we also get:

$$\begin{aligned} & |e^{i\theta_1} - e^{i\theta_2}|^2 \cdot |e^{i\theta_1} - e^{-i\theta_2}|^2 \\ &= 4(1 - \cos(\theta_1 - \theta_2))(1 - \cos(\theta_1 + \theta_2)) \\ &= 4(1 - \cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2)\cos(\theta_1 + \theta_2)) \\ &= 4 \left( 1 - 2\cos \theta_1 \cos \theta_2 + \frac{1}{2}(\cos 2\theta_1 + \cos 2\theta_2) \right). \end{aligned} \quad (\text{B.5})$$

Each cosine factor now only depends on one of the angles. Plugging (B.4) and (B.5) in (B.2) and using Fubini's theorem, we get:

$$c_4(\kappa) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2\kappa \cos \theta_1} \cdot h(\theta_1) \, d\theta_1, \quad (\text{B.6})$$

with:

$$h(\theta_1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2\kappa \cos \theta_2} \left( 1 + \frac{1}{2} \cos 2\theta_1 - 2\cos \theta_1 \cos \theta_2 + \frac{1}{2} \cos 2\theta_2 \right) d\theta_2. \quad (\text{B.7})$$

Now recalling the definition of the modified Bessel functions of the first kind [39],

$$I_\nu(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \cos \theta} \cos(\nu \theta) \, d\theta, \quad (\text{B.8})$$

we further simplify  $h$  to get:

$$h(\theta_1) = \left(1 + \frac{1}{2} \cos 2\theta_1\right) \cdot I_0(2\kappa) - 2 \cos \theta_1 \cdot I_1(2\kappa) + \frac{1}{2} \cdot I_2(2\kappa). \quad (\text{B.9})$$

Plugging (B.9) in (B.6) and resorting to Bessel functions again, we finally obtain the practical formula (4.8) for  $c_4(\kappa)$ :

$$\begin{aligned} c_4(\kappa) &= \left[ I_0(2\kappa) + \frac{1}{2} I_2(2\kappa) \right] \cdot I_0(2\kappa) - 2 I_1(2\kappa) \cdot I_1(2\kappa) + \frac{1}{2} I_0(2\kappa) \cdot I_2(2\kappa) \\ &= I_0(2\kappa)^2 - 2 I_1(2\kappa)^2 + I_0(2\kappa) I_2(2\kappa). \end{aligned} \quad (\text{B.10})$$

For generic  $n$ , the necessary manipulations are very similar to the developments in this appendix. For  $n = 2$  or  $3$ , the computations are easier. For  $n = 5$ , the computations take up about the same space. For  $n \geq 6$ , the same technique will still work but gets quite cumbersome.

In [11, Appendix A.6], Chikuse describes how the normalization coefficients for Langevin distributions on  $O(n)$  can be expressed in terms of hypergeometric functions with matrix arguments. One advantage of this method is that it generalizes to non-isotropic Langevin's. The method we demonstrated here, on the other hand, is tailored for our need (isotropic Langevin's on  $SO(n)$ ) and yields simple expressions in terms of Bessel functions—which are readily available in Matlab for example. Furthermore, this method applies well to compute the other integrals we need in this paper so that this appendix can be seen as an example.

## C Curvature terms

Cramér-Rao bounds for estimation problems on manifolds include extra terms in comparison to the classical CRB's on Euclidean spaces [7, 36, 41]. These terms stem from the possible curvature of the parameter space. As they can be quite tedious to compute, it is tempting to neglect curvature terms. One way to do this is to invoke high enough SNR's such that typical errors would verify  $\text{dist}^2(\mathbf{R}, \hat{\mathbf{R}}) \ll K_{\max}^{-1}$ , where  $K_{\max}$  is the largest sectional curvature of the parameter space [36]. Indeed, in this regime, error terms are small enough that curvature has essentially no effect. Unfortunately, for synchronization,  $K_{\max}^{-1}$  does not grow with  $N$ . Assuming we are interested in SNR's such that estimators commit a typical error on each  $\hat{R}_i$  of the order of, say, 1 degree, a typical  $\text{dist}^2(\mathbf{R}, \hat{\mathbf{R}})$  grows linearly with  $N$  (the number of rotations). The bound  $K_{\max}^{-1}$  remaining constant w.r.t.  $N$ , we would quickly reach values of  $N$  such that curvature cannot be legitimately neglected via this argument.

Consequently, we compute the curvature terms from theorems 4 and 5 in [7] for  $n = 2$  and  $n = 3$  explicitly. Our conclusion will be that it *is* legitimate to neglect them, at reasonable SNR's. This is mainly a consequence of the product nature of  $\mathcal{P}$ . Indeed, sectional curvatures on product manifolds vanish for most tangent 2-planes (more precisely, for all planes extending over more than one of the underlying terms of the product). Describing the curvature of such spaces only through a global bound  $K_{\max}$  hides this important structure. By investigating the curvature terms in detail, we capture the geometry accurately. For the same reason, we expect that curvature terms are negligible for  $n \geq 4$  too, but we do not conduct the calculations.

We first treat  $\mathcal{P}_A$  (3.14), then  $\mathcal{P}_\theta$  (3.18). We show that for rotations in the plane ( $n = 2$ ), the parameter spaces are flat, so that curvature terms vanish exactly. For rotations in space ( $n = 3$ ), we compute the curvature terms explicitly, and show that they are on the order of  $\mathcal{O}(\text{SNR}^{-2})$ , whereas dominant terms in the CRB are on the order of  $\mathcal{O}(\text{SNR}^{-1})$ .

### C.1 Curvature terms for $\mathcal{P}_A$

The manifold  $\mathcal{P}_A$  is a (product) Lie group. Hence, the Riemannian curvature tensor  $\mathcal{R}$  of  $\mathcal{P}_A$  on the tangent space  $T_{\mathbf{R}}\mathcal{P}_A$  is given by a simple formula [28, Corollary 11.10, p. 305]:

$$\langle \mathcal{R}(\mathbf{X}, \boldsymbol{\Omega}) \boldsymbol{\Omega}, \mathbf{X} \rangle = \frac{1}{4} \|\mathbf{X}, \boldsymbol{\Omega}\|^2, \quad (\text{C.1})$$

where  $[\mathbf{X}, \mathbf{\Omega}]$  is the Lie bracket of  $\mathbf{X} = (X_1, \dots, X_N)$  and  $\mathbf{\Omega} = (\Omega_1, \dots, \Omega_N)$ , two vectors (not necessarily orthonormal) in the tangent space  $\mathbf{T}_{\mathbf{R}}\mathcal{P}_A$ . Following [7, Theorem 4], in order to compute the curvature terms for the CRB of synchronization on  $\mathcal{P}_A$ , we first need to compute

$$\mathbf{R}_{\mathbf{m}}[\mathbf{\Omega}', \mathbf{\Omega}'] \triangleq \mathbb{E} \{ \langle \mathcal{R}(\mathbf{X}, P_A \mathbf{\Omega}') P_A \mathbf{\Omega}', \mathbf{X} \rangle \}, \quad (\text{C.2})$$

where  $\mathbf{\Omega}'$  is any tangent vector in  $\mathbf{T}_{\mathbf{R}}\mathcal{P}$  and  $P_A \mathbf{\Omega}'$  is its orthogonal projection on  $\mathbf{T}_{\mathbf{R}}\mathcal{P}_A$ . We expand  $\mathbf{X}$  and  $\mathbf{\Omega} = P_A \mathbf{\Omega}'$  using the orthonormal basis  $(\xi_{k\ell})_{k=1\dots N, \ell=1\dots d}$  (5.6) of  $\mathbf{T}_{\mathbf{R}}\mathcal{P} \supset \mathbf{T}_{\mathbf{R}}\mathcal{P}_A$ :

$$\mathbf{\Omega} = \sum_{k,\ell} \alpha_{k\ell} \xi_{k\ell} \quad \text{and} \quad \mathbf{X} = \sum_{k,\ell} \beta_{k\ell} \xi_{k\ell}, \quad (\text{C.3})$$

such that  $\Omega_k = R_k \sum_{\ell} \alpha_{k\ell} E_{\ell}$  and  $X_k = R_k \sum_{\ell} \beta_{k\ell} E_{\ell}$ . Of course,  $\alpha_{k\ell} = \beta_{k\ell} = 0 \ \forall k \in A$ . Then, since

$$[\mathbf{X}, \mathbf{\Omega}] = ([X_1, \Omega_1], \dots, [X_N, \Omega_N]),$$

it follows that:

$$\mathbf{R}_{\mathbf{m}}[\mathbf{\Omega}, \mathbf{\Omega}] = \mathbb{E} \left\{ \frac{1}{4} \|\mathbf{X}, \mathbf{\Omega}\|^2 \right\} = \mathbb{E} \left\{ \frac{1}{4} \sum_k \|[X_k, \Omega_k]\|^2 \right\} \quad (\text{C.4})$$

$$= \frac{1}{4} \sum_k \mathbb{E} \left\{ \left\| \sum_{\ell,s} \alpha_{k\ell} \beta_{ks} [E_{\ell}, E_s] \right\|^2 \right\}. \quad (\text{C.5})$$

For  $\mathbf{X}$  the tangent vector in  $\mathbf{T}_{\mathbf{R}}\mathcal{P}_A$  corresponding to the (random) estimation error  $\text{Log}_{\mathbf{R}}(\hat{\mathbf{R}})$ , the coefficients  $\beta_{k\ell}$  are random variables. The covariance matrix  $C_A$  (6.2) is given in terms of these coefficients by:

$$(C_A)_{kk', \ell\ell'} \triangleq \mathbb{E} \{ \langle \mathbf{X}, \xi_{k\ell} \rangle \langle \mathbf{X}, \xi_{k'\ell'} \rangle \} = \mathbb{E} \{ \beta_{k\ell} \beta_{k'\ell'} \}. \quad (\text{C.6})$$

The goal now is to express the entries of the matrix associated to  $\mathbf{R}_{\mathbf{m}}$  as linear combinations of the entries of  $C_A$ .

For  $n = 2$ , of course,  $\mathbf{R}_{\mathbf{m}} \equiv 0$  since Lie brackets vanish thanks to the commutativity of rotations in the plane. For  $n = 3$ , the special structure of  $\text{SO}(3)$ , namely its constant curvature, leads to nice expressions, which we obtain now.

For  $\text{SO}(3)$ , let us consider the orthonormal basis  $(E_1, E_2, E_3)$  of  $\mathfrak{so}(3)$  (5.5). Observe that it obeys  $[E_1, E_2] = E_3/\sqrt{2}$ ,  $[E_2, E_3] = E_1/\sqrt{2}$ ,  $[E_3, E_1] = E_2/\sqrt{2}$ . As a result, equation (C.5) simplifies and becomes:

$$\mathbf{R}_{\mathbf{m}}[\mathbf{\Omega}, \mathbf{\Omega}] = \frac{1}{8} \sum_k \mathbb{E} \{ (\alpha_{k2}\beta_{k3} - \alpha_{k3}\beta_{k2})^2 + (\alpha_{k3}\beta_{k1} - \alpha_{k1}\beta_{k3})^2 + (\alpha_{k1}\beta_{k2} - \alpha_{k2}\beta_{k1})^2 \}. \quad (\text{C.7})$$

We set out to compute the  $dN \times dN$  matrix  $R_m$  associated to the bi-linear operator  $\mathbf{R}_{\mathbf{m}}$  w.r.t. the basis (5.6). By definition,  $(R_m)_{kk', \ell\ell'} = \mathbf{R}_{\mathbf{m}}[\xi_{k\ell}, \xi_{k'\ell'}]$ . Equation C.7 readily yields the diagonal entries ( $k = k', \ell = \ell'$ ). Using polarization to determine off-diagonal entries [7, eq. (48)],

$$(R_m)_{kk', \ell\ell'} = \frac{1}{4} (\mathbf{R}_{\mathbf{m}}[\xi_{k\ell} + \xi_{k'\ell'}, \xi_{k\ell} + \xi_{k'\ell'}] - \mathbf{R}_{\mathbf{m}}[\xi_{k\ell} - \xi_{k'\ell'}, \xi_{k\ell} - \xi_{k'\ell'}]), \quad (\text{C.8})$$

it follows through simple calculations (taking into account the orthogonal projection onto  $\mathbf{T}_{\mathbf{R}}\mathcal{P}_A$  that appears in (C.2)) that:

$$(R_m)_{kk', \ell\ell'} = \begin{cases} \frac{1}{8} \sum_{s \neq \ell} (C_A)_{kk, ss} & \text{if } k = k' \notin A, \ell = \ell', \\ -\frac{1}{8} (C_A)_{kk, \ell\ell'} & \text{if } k = k' \notin A, \ell \neq \ell', \\ 0 & \text{otherwise.} \end{cases} \quad (\text{C.9})$$

Hence,  $R_m(C_A)$  is a block-diagonal matrix whose nonzero entries are linear functions of the entries of  $C_A$ . Theorem 4 in [7] requires (C.9) to compute the matrix  $R_m(F_A^\dagger)$ . Considering the special structure of the diagonal blocks of  $F_A^\dagger$  (6.5) (they are proportional to  $I_3$ ), we find that

$$R_m(F_A^\dagger) = \frac{1}{4} \text{ddiag}(F_A^\dagger) = \frac{3}{4} \text{ddiag}(\mathcal{L}_A^\dagger) \otimes I_3, \quad (\text{C.10})$$

where  $\text{ddiag}$  puts all off-diagonal entries of a matrix to zero. Thus, as the SNR goes up and hence as  $\mathcal{L}_A^\dagger$  goes down, the curvature term  $R_m(F_A^\dagger)F_A^\dagger + F_A^\dagger R_m(F_A^\dagger)$  in [7, Thm. 4] will become negligible compared to the main term in the CRB,  $F_A^\dagger$ .

## C.2 Curvature terms for $\mathcal{P}_\emptyset$

The manifold  $\mathcal{P}_\emptyset$  is a quotient manifold of  $\mathcal{P}$ . Hence, the Riemannian curvature tensor  $\mathcal{R}$  of  $\mathcal{P}_\emptyset$  is given by O'Neill's formula [28, Thm 7.47, p. 213 and Lemma 3.39, p. 77], showing that the quotient operation can only increase the curvature of the parameter space:

$$\langle \mathcal{R}(\text{D}\pi \mathbf{X}, \text{D}\pi \mathbf{\Omega}) \text{D}\pi \mathbf{\Omega}, \text{D}\pi \mathbf{X} \rangle = \frac{1}{4} \|\mathbf{X}, \mathbf{\Omega}\|^2 + \frac{3}{4} \|\mathbf{X}, \mathbf{\Omega}\|^\mathcal{V}\|^2, \quad (\text{C.11})$$

where  $\mathbf{X}, \mathbf{\Omega}$  are horizontal vectors in  $\text{H}_\mathbf{R} \subset \text{T}_\mathbf{R}\mathcal{P}$  associated to tangent vectors to  $\mathcal{P}_\emptyset$  via the differential of the Riemannian submersion  $\text{D}\pi(\mathbf{R})$  (3.20), denoted simply as  $\text{D}\pi$  for convenience. The vector  $[\mathbf{X}, \mathbf{\Omega}]^\mathcal{V} \in \text{V}_\mathbf{R} \subset \text{T}_\mathbf{R}\mathcal{P}$  is the vertical part of  $[\mathbf{X}, \mathbf{\Omega}]$ , i.e., the component that is parallel to the fibers. Since in our case, moving along a fiber consists in changing all rotations along the same direction,  $[\mathbf{X}, \mathbf{\Omega}]^\mathcal{V}$  corresponds to the mean component of  $[\mathbf{X}, \mathbf{\Omega}]$ :

$$[\mathbf{X}, \mathbf{\Omega}]^\mathcal{V} = (R_1\omega, \dots, R_N\omega), \text{ with } \omega = \frac{1}{N} \sum_{k=1}^N [R_k^\top X_k, R_k^\top \Omega_k]. \quad (\text{C.12})$$

For  $n = 2$ , since  $[\mathbf{X}, \mathbf{\Omega}] = 0$ ,  $[\mathbf{X}, \mathbf{\Omega}]^\mathcal{V} = 0$  also, hence  $\mathcal{P}_\emptyset$  is still a flat manifold, despite the quotient operation. We now show that for  $n = 3$  the curvature terms in the CRB [7, Thm. 5] are equivalent to the curvature terms for  $\mathcal{P}_A$  with  $A := \emptyset$  plus extra terms that decay as  $1/N$  and can thus be neglected.

The curvature operator  $\mathbf{R}_\mathbf{m}$  [7, eq. (54)] is given by:

$$\mathbf{R}_\mathbf{m}[\xi_{k\ell}, \xi_{k\ell}] \triangleq \mathbb{E} \{ \langle \mathcal{R}(\text{D}\pi \mathbf{X}, \text{D}\pi \xi_{k\ell}) \text{D}\pi \xi_{k\ell}, \text{D}\pi \mathbf{X} \rangle \} \quad (\text{C.13})$$

$$= \mathbb{E} \left\{ \frac{1}{4} \|\mathbf{X}, \xi_{k\ell} - \xi_{k\ell}^\mathcal{V}\|^2 + \frac{3}{4} \|\mathbf{X}, \xi_{k\ell} - \xi_{k\ell}^\mathcal{V}\|^\mathcal{V}\|^2 \right\}. \quad (\text{C.14})$$

The tangent vector  $\xi_{k\ell} - \xi_{k\ell}^\mathcal{V}$  is, by construction, the horizontal part of  $\xi_{k\ell}$ . The vertical part decreases in size as  $N$  grows:

$$\|\xi_{k\ell}^\mathcal{V}\|^2 = \left\| \frac{1}{N} (R_1 E_\ell, \dots, R_N E_\ell) \right\|^2 = \frac{1}{N}. \quad (\text{C.15})$$

It follows that:

$$\mathbb{E} \{ \|\mathbf{X}, \xi_{k\ell} - \xi_{k\ell}^\mathcal{V}\|^2 \} = \mathbb{E} \{ \|\mathbf{X}, \xi_{k\ell}\|^2 \} + \mathcal{O}(1/N). \quad (\text{C.16})$$

Hence, up to a factor that decays as  $1/N$ , the first term in the curvature operator  $\mathbf{R}_\mathbf{m}$  is the same as that of the previous section for  $\mathcal{P}_A$ , with  $A := \emptyset$ . We now deal with the second term defining  $\mathbf{R}_\mathbf{m}$ :

$$[\mathbf{X}, \xi_{k\ell}]^\mathcal{V} = (R_1\omega, \dots, R_N\omega), \text{ with} \quad (\text{C.17})$$

$$\omega = \frac{1}{N} [R_k^\top X_k, E_\ell] = \frac{1}{N} \sum_s \beta_{ks} [E_s, E_\ell]. \quad (\text{C.18})$$

It is now clear that for large  $N$  this second term is negligible compared to  $\mathbb{E} \{ \|\mathbf{X}, \boldsymbol{\xi}_{k\ell}\|^2 \}$ :

$$\|[\mathbf{X}, \boldsymbol{\xi}_{k\ell}]^\vee\|^2 = N\|\omega\|^2 = \mathcal{O}(1/N). \quad (\text{C.19})$$

Applying polarization to  $\mathbf{R}_m$  to compute off-diagonal terms then concludes the argument showing that the curvature terms in the CRB for synchronization of rotations on  $\mathcal{P}_\emptyset$ , despite an increased curvature tensor owing to the quotient operation (C.11), are very close (within a  $\mathcal{O}(1/N)$  term) to the curvature terms established earlier for synchronization on  $\mathcal{P}_A$ , with  $A := 0$ . We do not include an exact derivation of these terms as it is quite lengthy and does not bring much insight to the problem.

## D Proof that $G_{ij}(QQZQ^\top) = QG_{ij}(Z)Q^\top$ and that $G_{ij}(Z^\top) = -G_{ij}(Z)$ .

Recall the definition of  $G_{ij} : \text{SO}(n) \rightarrow \mathfrak{so}(n)$  (5.3):

$$G_{ij}(Z) = [\text{grad log } f_{ij}(Z)]^\top Z. \quad (\text{D.1})$$

Now let us introduce a few functions:

$$g : \text{SO}(n) \rightarrow \mathbb{R} : Z \mapsto g(Z) = \log f_{ij}(Z), \quad (\text{D.2})$$

$$h_1 : \text{SO}(n) \rightarrow \text{SO}(n) : Z \mapsto h_1(Z) = QQZQ^\top, \quad (\text{D.3})$$

$$h_2 : \text{SO}(n) \rightarrow \text{SO}(n) : Z \mapsto h_2(Z) = Z^\top. \quad (\text{D.4})$$

Notice that because of Assumption 3 ( $f_{ij}$  is only a function of the eigenvalues of its argument), we have  $g \circ h_i \equiv g$  for  $i = 1, 2$ . Hence,

$$\text{grad } g(Z) = \text{grad}(g \circ h_i)(Z) = (Dh_i(Z))^* [\text{grad } g(h_i(Z))], \quad (\text{D.5})$$

where  $(Dh_i(Z))^*$  denotes the adjoint of the differential  $Dh_i(Z)$ , defined by

$$\forall H_1, H_2 \in \text{T}_Z \text{SO}(n), \quad \langle Dh_i(Z)[H_1], H_2 \rangle = \langle H_1, (Dh_i(Z))^*[H_2] \rangle. \quad (\text{D.6})$$

The rightmost equality of (D.5) follows from the chain rule. Indeed, starting with the definition of gradient, we have:

$$\begin{aligned} \forall H \in \text{T}_Z \text{SO}(n), \quad \langle \text{grad}(g \circ h_i)(Z), H \rangle &= D(g \circ h_i)(Z)[H] \\ &= Dg(h_i(Z))[Dh_i(Z)[H]] \\ &= \langle \text{grad } g(h_i(Z)), Dh_i(Z)[H] \rangle \\ &= \langle (Dh_i(Z))^* [\text{grad } g(h_i(Z))], H \rangle. \end{aligned} \quad (\text{D.7})$$

Let us compute the differentials of the  $h_i$ 's and their adjoints:

$$Dh_1(Z)[H] = QHQ^\top, \quad (Dh_1(Z))^*[H] = Q^\top H Q, \quad (\text{D.8})$$

$$Dh_2(Z)[H] = H^\top, \quad (Dh_2(Z))^*[H] = H^\top. \quad (\text{D.9})$$

Plugging this in (D.5), we find two identities (one for  $h_1$  and one for  $h_2$ ):

$$\text{grad log } f_{ij}(Z) = Q^\top [\text{grad log } f_{ij}(QQZQ^\top)] Q, \quad (\text{D.10})$$

$$\text{grad log } f_{ij}(Z) = [\text{grad log } f_{ij}(Z^\top)]^\top. \quad (\text{D.11})$$

The desired result about the  $G_{ij}$ 's now follows easily. For any  $Q \in \text{O}(n)$ ,

$$G_{ij}(QQZQ^\top) = [\text{grad log } f_{ij}(QQZQ^\top)]^\top QQZQ^\top = [Q \text{grad log } f_{ij}(Z) Q^\top]^\top QQZQ^\top = QG_{ij}(Z)Q^\top. \quad (\text{D.12})$$

And similarly:

$$G_{ij}(Z^\top) = [\text{grad log } f_{ij}(Z^\top)]^\top Z^\top = \text{grad log } f_{ij}(Z) Z^\top = ZG_{ij}^\top(Z)Z^\top = -ZG_{ij}(Z)Z^\top = -G_{ij}(Z),$$

where we used that  $G_{ij}(Z)$  is skew-symmetric and we used (D.12) for the rightmost equality.

## E Proof of lemma 5.2

*Proof.* We give a constructive proof, distinguishing among three cases. (*Case 1:*  $\{i, j\} \cap \{k, \ell\} = \emptyset$ ). Construct  $T$  as the identity  $I_n$  with columns  $i$  and  $k$  swapped, as well as columns  $j$  and  $\ell$ . Construct  $S$  as  $I_n$  with  $S_{ii} := -1$ . By construction, it holds that  $T^\top ET = E'$ ,  $T^\top E'T = E$ ,  $SES = -E$  and  $SE'S = E'$ . Set  $P = TS$  to conclude:  $P^\top EP = ST^\top ETS = SE'S = E'$ ,  $P^\top E'P = ST^\top E'TS = SES = -E$ . (*Case 2:*  $i = k, j \neq \ell$ ). Construct  $T$  as the identity  $I_n$  with columns  $j$  and  $\ell$  swapped. Construct  $S$  as  $I_n$  with  $S_{jj} := -1$ . The same properties will hold. Set  $P = TS$  to conclude. (*Case 3:*  $i = \ell, j \neq k$ ). Construct  $T$  as the identity  $I_n$  with columns  $j$  and  $k$  swapped and with  $T_{ii} := -1$ . Construct  $S$  as  $I_n$  with  $S_{jj} := -1$ . Set  $P = TS$  to conclude. (*Cases 4 and 5:*  $j = k$  or  $j = \ell$ ). The same construction goes through.  $\square$

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